CRYSTALS OF FOCK SPACES AND CYCLOTOMIC RATIONAL DOUBLE AFFINE HECKE ALGEBRAS

PENG SHAN

Abstract. We define the i-restriction and i-induction functors on the category $\mathcal{O}$ of the cyclotomic rational double affine Hecke algebras. This yields a crystal on the set of isomorphism classes of simple modules, which is isomorphic to the crystal of a Fock space.

Résumé. On définit les foncteurs de i-restriction et i-induction sur la catégorie $\mathcal{O}$ des algèbres de Hecke doublement affine rationnelles cyclotomiques. Ceci donne lieu à un cristal sur l'ensemble des classes d'isomorphismes de modules simples, qui est isomorphe au cristal d'un espace de Fock.

Introduction

In [A1], S. Ariki defined the i-restriction and i-induction functors for cyclotomic Hecke algebras. He showed that the Grothendieck group of the category of finitely generated projective modules of these algebras admits a module structure over the affine Lie algebra of type $A^{(1)}$, with the action of Chevalley generators given by the i-restriction and i-induction functors.

The restriction and induction functors for rational DAHA's(=double affine Hecke algebras) were recently defined by R. Bezrukavnikov and P. Etingof. With these functors, we give an analogue of Ariki's construction for the category $\mathcal{O}$ of cyclotomic rational DAHA's: we show that as a module over the type $A^{(1)}$ affine Lie algebra, the Grothendieck group of this category is isomorphic to a Fock space. We also construct a crystal on the set of isomorphism classes of simple modules in the category $\mathcal{O}$. It is isomorphic to the crystal of the Fock space. Recall that this Fock space also enters in some conjectural description of the decomposition numbers for the category $\mathcal{O}$ considered here. See [U], [Y], [R1] for related works.

Notation

For A an algebra, we will write A-mod for the category of finitely generated A-modules. For $f : A \to B$ an algebra homomorphism from A to another algebra B such that B is finitely generated over A, we will write

$$f_* : B\text{-mod} \to A\text{-mod}$$

for the restriction functor and we write

$$f^* : A\text{-mod} \to B\text{-mod}, \quad M \mapsto B \otimes_A M.$$

A $\mathbb{C}$-linear category $\mathcal{A}$ is called artinian if the Hom sets are finite dimensional $\mathbb{C}$-vector spaces and every object has a finite length. Given an object M in $\mathcal{A}$, we denote by $\mathrm{soc}(M)$ (resp. $\mathrm{head}(M)$) the socle (resp. the head) of M, which is the largest semi-simple subobject (quotient) of M.

Let $\mathcal{C}$ be an abelian category. The Grothendieck group of $\mathcal{C}$ is the quotient of the free abelian group generated by objects in $\mathcal{C}$ modulo the relations $M = M' + M''$

Titre en français. Cristaux d'espaces de Fock et algèbres de Hecke doublement affine rationnelles cyclotomiques.

for all objects M, M', M'' in $\mathcal{C}$ such that there is an exact sequence $0 \to M' \to M \to M'' \to 0$. Let $K(\mathcal{C})$ denote the complexified Grothendieck group, a $\mathbb{C}$-vector space. For each object M in $\mathcal{C}$, let $[M]$ be its class in $K(\mathcal{C})$. Any exact functor $F : \mathcal{C} \to \mathcal{C}'$ between two abelian categories induces a vector space homomorphism $K(\mathcal{C}) \to K(\mathcal{C}')$, which we will denote by F again. Given an algebra A we will abbreviate $K(A) = K(A\text{-mod})$.

Denote by $\mathrm{Fct}(\mathcal{C}, \mathcal{C}')$ the category of functors from a category $\mathcal{C}$ to a category $\mathcal{C}'$. For $F \in \mathrm{Fct}(\mathcal{C}, \mathcal{C}')$ write $\mathrm{End}(F)$ for the ring of endomorphisms of the functor F. We denote by $1_F : F \to F$ the identity element in $\mathrm{End}(F)$. Let $G \in \mathrm{Fct}(\mathcal{C}', \mathcal{C}'')$ be a functor from $\mathcal{C}'$ to another category $\mathcal{C}''$. For any $X \in \mathrm{End}(F)$ and any $X' \in \mathrm{End}(G)$ we write $X'X : G \circ F \to G \circ F$ for the morphism of functors given by $X'X(M) = X'(F(M)) \circ G(X(M))$ for any $M \in \mathcal{C}$.

Let $e \geqslant 2$ be an integer and z be a formal parameter. Denote by $\mathfrak{sl}_e$ the Lie algebra of traceless $e \times e$ complex matrices. Write E_{ij} for the elementary matrix with 1 in the position (i, j) and 0 elsewhere. The type $A^{(1)}$ affine Lie algebra $\widehat{\mathfrak{sl}}_e$ is $\mathfrak{sl}_e \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}c$ with c a central element. The Lie bracket is the usual one. We will denote the Chevalley generators of $\widehat{\mathfrak{sl}}_e$ as follows:

$$\begin{aligned} e_i &= E_{i,i+1} \otimes 1, \quad & f_i &= F_{i+1,i} \otimes 1, \quad & h_i &= (E_{ii} - E_{i+1,i+1}) \otimes 1, \quad 1 \leqslant i \leqslant e-1, \\ e_0 &= E_{e1} \otimes z, & f_0 &= E_{1e} \otimes z^{-1}, & h_0 &= (E_{ee} - E_{11}) \otimes 1 + c. \end{aligned}$$

For $i \in \mathbb{Z}/e\mathbb{Z}$ we will denote the simple root (resp. coroot) corresponding to e_i by α_i (resp. $\alpha_i^\vee$). The fundamental weights are $\{\Lambda_i : i \in \mathbb{Z}/e\mathbb{Z}\}$ with $\alpha_i^\vee(\Lambda_j) = \delta_{ij}$ for any $i, j \in \mathbb{Z}/e\mathbb{Z}$. We will write P for the weight lattice, the free abelian group generated by the fundamental weights.

1. Reminders on Hecke algebras, rational DAHA's and restriction functors

1.1. **Hecke algebras.** Let $\mathfrak{h}$ be a finite dimensional vector space over $\mathbb{C}$. Recall that a pseudo-reflection is a non trivial element s of $GL(\mathfrak{h})$ which acts trivially on a hyperplane, called the reflecting hyperplane of s. Let $W \subset GL(\mathfrak{h})$ be a finite subgroup generated by pseudo-reflections. Let $\mathcal{S}$ be the set of pseudo-reflections in W and $\mathcal{A}$ be the set of reflecting hyperplanes. We set $\mathfrak{h}_{reg} = \mathfrak{h} - \bigcup_{H \in \mathcal{A}} H$, it is stable under the action of W. Fix $x_0 \in \mathfrak{h}_{reg}$ and identify it with its image in $\mathfrak{h}_{reg}/W$. By definition the braid group attached to $(W, \mathfrak{h})$, denoted by $B(W, \mathfrak{h})$, is the fundamental group $\pi_1(\mathfrak{h}_{reg}/W, x_0)$.

For any $H \in \mathcal{A}$, let W_H be the pointwise stabilizer of H. This is a cyclic group. Write e_H for the order of W_H. Let s_H be the unique element in W_H whose determinant is $\exp(\frac{2\pi\sqrt{-1}}{e_H})$. Let q be a map from $\mathcal{S}$ to $\mathbb{C}^*$ that is constant on the W-conjugacy classes. Following [BMR, Definition 4.21] the Hecke algebra $\mathscr{H}_q(W, \mathfrak{h})$ attached to $(W, \mathfrak{h})$ with parameter q is the quotient of the group algebra $\mathbb{C}B(W, \mathfrak{h})$ by the relations:

$$(T_{s_H} - 1) \prod_{t \in W_H \cap \mathcal{S}} (T_{s_H} - q(t)) = 0, \quad H \in \mathcal{A}. \tag{1.1}$$

Here T_{s_H} is a generator of the monodromy around H in $\mathfrak{h}_{reg}/W$ such that the lift of T_{s_H} in $\pi_1(W, \mathfrak{h}_{reg})$ via the map $\mathfrak{h}_{reg} \to \mathfrak{h}_{reg}/W$ is represented by a path from x_0 to $s_H(x_0)$. See [BMR, Section 2B] for a precise definition. When the subspace $\mathfrak{h}^W$ of fixed points of W in $\mathfrak{h}$ is trivial, we abbreviate

$$B_W = B(W, \mathfrak{h}), \quad \mathscr{H}_q(W) = \mathscr{H}_q(W, \mathfrak{h}).$$

1.2. **Parabolic restriction and induction for Hecke algebras.** In this section we will assume that $\mathfrak{h}^W = 1$. A parabolic subgroup W' of W is by definition the stabilizer of a point $b \in \mathfrak{h}$. By a theorem of Steinberg, the group W' is also generated by pseudo-reflections. Let q' be the restriction of q to $\mathcal{S}' = W' \cap \mathcal{S}$. There is an explicit inclusion $\imath_q : \mathscr{H}_{q'}(W') \hookrightarrow \mathscr{H}_q(W)$ given by [BMR, Section 2D]. The restriction functor

$$^{\mathscr{H}}\mathrm{Res}^W_{W'} : \mathscr{H}_q(W)\text{-mod} \to \mathscr{H}_{q'}(W')\text{-mod}$$

is the functor $(\imath_q)_*$. The induction functor

$$^{\mathscr{H}}\mathrm{Ind}^W_{W'} = \mathscr{H}_q(W) \otimes_{\mathscr{H}_{q'}(W')} -$$

is left adjoint to $^{\mathscr{H}}\mathrm{Res}^W_{W'}$. The coinduction functor

$$^{\mathscr{H}}\mathrm{coInd}^W_{W'} = \mathrm{Hom}_{\mathscr{H}_{q'}(W')}(\mathscr{H}_q(W), -)$$

is right adjoint to $^{\mathscr{H}}\mathrm{Res}^W_{W'}$. The three functors above are all exact.

Let us recall the definition of $\imath_q$. It is induced from an inclusion $\imath : B_{W'} \hookrightarrow B_W$, which is in turn the composition of three morphisms ℓ, κ, $\jmath$ defined as follows. First, let $\mathcal{A}' \subset \mathcal{A}$ be the set of reflecting hyperplanes of W'. Write

$$\overline{\mathfrak{h}} = \mathfrak{h}/\mathfrak{h}^{W'}, \quad \overline{\mathcal{A}} = \{\overline{H} = H/\mathfrak{h}^{W'} \,:\, H \in \mathcal{A}'\}, \quad \overline{\mathfrak{h}}_{reg} = \overline{\mathfrak{h}} - \bigcup_{\overline{H} \in \overline{\mathcal{A}}} \overline{H}, \quad \mathfrak{h}'_{reg} = \mathfrak{h} - \bigcup_{H \in \mathcal{A}'} H.$$

The canonical epimorphism $p : \mathfrak{h} \to \overline{\mathfrak{h}}$ induces a trivial W'-equivariant fibration $p : \mathfrak{h}'_{reg} \to \overline{\mathfrak{h}}_{reg}$, which yields an isomorphism

$$\ell : B_{W'} = \pi_1(\overline{\mathfrak{h}}_{reg}/W', p(x_0)) \xrightarrow{\sim} \pi_1(\mathfrak{h}'_{reg}/W', x_0). \tag{1.2}$$

Endow $\mathfrak{h}$ with a W-invariant hermitian scalar product. Let $||\cdot||$ be the associated norm. Set

$$\Omega = \{x \in \mathfrak{h} \,:\, ||x - b|| < \varepsilon\}, \tag{1.3}$$

where ε is a positive real number such that the closure of Ω does not intersect any hyperplane that is in the complement of $\mathcal{A}'$ in $\mathcal{A}$. Let $\gamma : [0,1] \to \mathfrak{h}$ be a path such that $\gamma(0) = x_0$, $\gamma(1) = b$ and $\gamma(t) \in \mathfrak{h}_{reg}$ for $0 < t < 1$. Let $u \in [0,1[$ such that $x_1 = \gamma(u)$ belongs to Ω, write γ_u for the restriction of γ to $[0,u]$. Consider the homomorphism

$$\sigma : \pi_1(\Omega \cap \mathfrak{h}_{reg}, x_1) \to \pi_1(\mathfrak{h}_{reg}, x_0), \quad \lambda \mapsto \gamma_u^{-1} \cdot \lambda \cdot \gamma_u.$$

The canonical inclusion $\mathfrak{h}_{reg} \hookrightarrow \mathfrak{h}'_{reg}$ induces a homomorphism $\pi_1(\mathfrak{h}_{reg}, x_0) \to \pi_1(\mathfrak{h}'_{reg}, x_0)$. Composing it with σ gives an invertible homomorphism

$$\pi_1(\Omega \cap \mathfrak{h}_{reg}, x_1) \to \pi_1(\mathfrak{h}'_{reg}, x_0).$$

Since Ω is W'-invariant, its inverse gives an isomorphism

$$\kappa : \pi_1(\mathfrak{h}'_{reg}/W', x_0) \xrightarrow{\sim} \pi_1((\Omega \cap \mathfrak{h}_{reg})/W', x_1). \tag{1.4}$$

Finally, we see from above that σ is injective. So it induces an inclusion

$$\pi_1((\Omega \cap \mathfrak{h}_{reg})/W', x_1) \hookrightarrow \pi_1(\mathfrak{h}_{reg}/W', x_0).$$

Composing it with the canonical inclusion $\pi_1(\mathfrak{h}_{reg}/W', x_0) \hookrightarrow \pi_1(\mathfrak{h}_{reg}/W, x_0)$ gives an injective homomorphism

$$\jmath : \pi_1((\Omega \cap \mathfrak{h}_{reg})/W', x_1) \hookrightarrow \pi_1(\mathfrak{h}_{reg}/W, x_0) = B_W. \tag{1.5}$$

By composing ℓ, κ, $\jmath$ we get the inclusion

$$\imath = \jmath \circ \kappa \circ \ell : B_{W'} \hookrightarrow B_W. \tag{1.6}$$

It is proved in [BMR, Section 4C] that $\imath$ preserves the relations in (1.1). So it induces an inclusion of Hecke algebras which is the desired inclusion

$$\imath_q : \mathscr{H}_{q'}(W') \hookrightarrow \mathscr{H}_q(W).$$

For $\imath$, $\imath' : B_{W'} \hookrightarrow B_W$ two inclusions defined as above via different choices of the path γ, there exists an element $\rho \in P_W = \pi_1(\mathfrak{h}_{reg}, x_0)$ such that for any $a \in B_{W'}$ we have $\imath(a) = \rho\imath'(a)\rho^{-1}$. In particular, the functors $\imath_*$ and $(\imath')_*$ from B_W-mod to $B_{W'}$-mod are isomorphic. Also, we have $(\imath_q)_* \cong (\imath'_q)_*$. So there is a unique restriction functor ${}^{\mathscr{H}}\mathrm{Res}^W_{W'}$ up to isomorphisms.

1.3. **Rational DAHA's.** Let c be a map from $\mathcal{S}$ to $\mathbb{C}$ that is constant on the W-conjugacy classes. The rational DAHA attached to W with parameter c is the quotient $H_c(W, \mathfrak{h})$ of the smash product of $\mathbb{C}W$ and the tensor algebra of $\mathfrak{h} \oplus \mathfrak{h}^*$ by the relations

$$[x, x'] = 0, \quad [y, y'] = 0, \quad [y, x] = \langle x, y\rangle - \sum_{s\in\mathcal{S}} c_s\langle\alpha_s, y\rangle\langle x, \alpha_s^\vee\rangle s,$$

for all $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$. Here $\langle\cdot,\cdot\rangle$ is the canonical pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$, the element α_s is a generator of $\mathrm{Im}(s|_{\mathfrak{h}^*} - 1)$ and $\alpha_s^\vee$ is the generator of $\mathrm{Im}(s|_{\mathfrak{h}} - 1)$ such that $\langle\alpha_s, \alpha_s^\vee\rangle = 2$.

For $s \in \mathcal{S}$ write λ_s for the non trivial eigenvalue of s in $\mathfrak{h}^*$. Let $\{x_i\}$ be a basis of $\mathfrak{h}^*$ and let $\{y_i\}$ be the dual basis. Let

$$\mathbf{eu} = \sum_i x_i y_i + \frac{\dim(\mathfrak{h})}{2} - \sum_{s\in\mathcal{S}} \frac{2c_s}{1-\lambda_s} s \tag{1.7}$$

be the Euler element in $H_c(W, \mathfrak{h})$. Its definition is independent of the choice of the basis $\{x_i\}$. We have

$$[\mathbf{eu}, x_i] = x_i, \quad [\mathbf{eu}, y_i] = -y_i, \quad [\mathbf{eu}, s] = 0. \tag{1.8}$$

1.4. The category $\mathcal{O}$ of $H_c(W, \mathfrak{h})$ is the full subcategory $\mathcal{O}_c(W, \mathfrak{h})$ of the category of $H_c(W, \mathfrak{h})$-modules consisting of objects that are finitely generated as $\mathbb{C}[\mathfrak{h}]$-modules and $\mathfrak{h}$-locally nilpotent. We recall from [GGOR, Section 3] the following properties of $\mathcal{O}_c(W, \mathfrak{h})$.

The action of the Euler element $\mathbf{eu}$ on a module in $\mathcal{O}_c(W, \mathfrak{h})$ is locally finite. The category $\mathcal{O}_c(W, \mathfrak{h})$ is a highest weight category. In particular, it is artinian. Write $\mathrm{Irr}(W)$ for the set of isomorphism classes of irreducible representations of W. The poset of standard modules in $\mathcal{O}_c(W, \mathfrak{h})$ is indexed by $\mathrm{Irr}(W)$ with the partial order given by [GGOR, Theorem 2.19]. More precisely, for $\xi \in \mathrm{Irr}(W)$, equip it with a $\mathbb{C}W \ltimes \mathbb{C}[\mathfrak{h}^*]$-module structure by letting the elements in $\mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]$ act by zero, the standard module corresponding to ξ is

$$\Delta(\xi) = H_c(W, \mathfrak{h}) \otimes_{\mathbb{C}W\ltimes\mathbb{C}[\mathfrak{h}^*]} \xi.$$

It is an indecomposable module with a simple head $L(\xi)$. The set of isomorphism classes of simple modules in $\mathcal{O}_c(W, \mathfrak{h})$ is

$$\{[L(\xi)] : \xi \in \mathrm{Irr}(W)\}.$$

It is a basis of the $\mathbb{C}$-vector space $K(\mathcal{O}_c(W, \mathfrak{h}))$. The set $\{[\Delta(\xi)] : \xi \in \mathrm{Irr}(W)\}$ gives another basis of $K(\mathcal{O}_c(W, \mathfrak{h}))$.

We say a module N in $\mathcal{O}_c(W, \mathfrak{h})$ has a standard filtration if it admits a filtration

$$0 = N_0 \subset N_1 \subset \ldots \subset N_n = N$$

such that each quotient N_i/N_{i-1} is isomorphic to a standard module. We denote by $\mathcal{O}_c^\Delta(W, \mathfrak{h})$ the full subcategory of $\mathcal{O}_c(W, \mathfrak{h})$ consisting of such modules.

Lemma 1.1. *(1)Any projective object in $\mathcal{O}_c(W,\mathfrak{h})$ has a standard filtration.*

(2)A module in $\mathcal{O}_c(W,\mathfrak{h})$ has a standard filtration if and only if it is free as a $\mathbb{C}[\mathfrak{h}]$-module.

Both (1) and (2) are given by [GGOR, Proposition 2.21].

The category $\mathcal{O}_c(W,\mathfrak{h})$ has enough projective objects and has finite homological dimension [GGOR, Section 4.3.1]. In particular, any module in $\mathcal{O}_c(W,\mathfrak{h})$ has a finite projective resolution. Write $\mathrm{Proj}_c(W,\mathfrak{h})$ for the full subcategory of projective modules in $\mathcal{O}_c(W,\mathfrak{h})$. Let

$$I:\mathrm{Proj}_c(W,\mathfrak{h})\to\mathcal{O}_c(W,\mathfrak{h})$$

be the canonical embedding functor. We have the following lemma.

Lemma 1.2. *For any abelian category $\mathcal{A}$ and any right exact functors F_1, F_2 from $\mathcal{O}_c(W,\mathfrak{h})$ to $\mathcal{A}$, the homomorphism of vector spaces*

$$r_I:\mathrm{Hom}(F_1,F_2)\to\mathrm{Hom}(F_1\circ I,F_2\circ I),\quad \gamma\mapsto\gamma 1_I$$

is an isomorphism.

In particular, if the functor $F_1\circ I$ is isomorphic to $F_2\circ I$, then we have $F_1\cong F_2$.

Proof. We need to show that for any morphism of functors $\nu:F_1\circ I\to F_2\circ I$ there is a unique morphism $\tilde{\nu}:F_1\to F_2$ such that $\tilde{\nu}1_I=\nu$. Since $\mathcal{O}_c(W,\mathfrak{h})$ has enough projectives, for any $M\in\mathcal{O}_c(W,\mathfrak{h})$ there exists P_0, P_1 in $\mathrm{Proj}_c(W,\mathfrak{h})$ and an exact sequence in $\mathcal{O}_c(W,\mathfrak{h})$

$$P_1\xrightarrow{d_1}P_0\xrightarrow{d_0}M\longrightarrow 0. \tag{1.9}$$

Applying the right exact functors F_1, F_2 to this sequence we get the two exact sequences in the diagram below. The morphism of functors $\nu:F_1\circ I\to F_2\circ I$ yields well defined morphisms $\nu(P_1)$, $\nu(P_0)$ such that the square commutes

$$\begin{tikzcd}
F_1(P_1) \arrow[r, "F_1(d_1)"] \arrow[d, "\nu(P_1)"] & F_1(P_0) \arrow[r, "F_1(d_0)"] \arrow[d, "\nu(P_0)"] & F_1(M) \arrow[r] & 0 \\
F_2(P_1) \arrow[r, "F_2(d_1)"] & F_2(P_0) \arrow[r, "F_2(d_0)"] & F_2(M) \arrow[r] & 0.
\end{tikzcd}$$

Define $\tilde{\nu}(M)$ to be the unique morphism $F_1(M)\to F_2(M)$ that makes the diagram commute. Its definition is independent of the choice of P_0, P_1, and it is independent of the choice of the exact sequence (1.9). The assignment $M\mapsto\tilde{\nu}(M)$ gives a morphism of functor $\tilde{\nu}:F_1\to F_2$ such that $\tilde{\nu}1_I=\nu$. It is unique by the uniqueness of the morphism $\tilde{\nu}(M)$. ☐

1.5. **KZ functor.** The Knizhnik-Zamolodchikov functor is an exact functor from the category $\mathcal{O}_c(W,\mathfrak{h})$ to the category $\mathscr{H}_q(W,\mathfrak{h})$-mod, where q is a certain parameter associated with c. Let us recall its definition from [GGOR, Section 5.3].

Let $\mathcal{D}(\mathfrak{h}_{reg})$ be the algebra of differential operators on $\mathfrak{h}_{reg}$. Write

$$H_c(W,\mathfrak{h}_{reg})=H_c(W,\mathfrak{h})\otimes_{\mathbb{C}[\mathfrak{h}]}\mathbb{C}[\mathfrak{h}_{reg}].$$

We consider the Dunkl isomorphism, which is an isomorphism of algebras

$$H_c(W,\mathfrak{h}_{reg})\xrightarrow{\sim}\mathcal{D}(\mathfrak{h}_{reg})\rtimes\mathbb{C}W$$

given by $x\mapsto x$, $w\mapsto w$ for $x\in\mathfrak{h}^*$, $w\in W$, and

$$y\mapsto\partial_y+\sum_{s\in\mathcal{S}}\frac{2c_s}{1-\lambda_s}\frac{\alpha_s(y)}{\alpha_s}(s-1),\quad\text{for } y\in\mathfrak{h}.$$

For any $M\in\mathcal{O}_c(W,\mathfrak{h})$, write

$$M_{\mathfrak{h}_{reg}}=M\otimes_{\mathbb{C}[\mathfrak{h}]}\mathbb{C}[\mathfrak{h}_{reg}].$$

It identifies via the Dunkl isomorphism with a $\mathcal{D}(\mathfrak{h}_{reg})\rtimes W$-module which is finitely generated over $\mathbb{C}[\mathfrak{h}_{reg}]$. Hence $M_{\mathfrak{h}_{reg}}$ is a W-equivariant vector bundle on $\mathfrak{h}_{reg}$ with an integrable connection ∇ given by $\nabla_y(m)=\partial_y m$ for $m\in M$, $y\in\mathfrak{h}$. It is proved in [GGOR, Proposition 5.7] that the connection ∇ has regular singularities. Now, regard $\mathfrak{h}_{reg}$ as a complex manifold endowed with the transcendental topology. Denote by $\mathcal{O}^{an}_{\mathfrak{h}_{reg}}$ the sheaf of holomorphic functions on $\mathfrak{h}_{reg}$. For any free $\mathbb{C}[\mathfrak{h}_{reg}]$-module N of finite rank, we consider

$$N^{an}=N\otimes_{\mathbb{C}[\mathfrak{h}_{reg}]}\mathcal{O}^{an}_{\mathfrak{h}_{reg}}.$$

It is an analytic locally free sheaf on $\mathfrak{h}_{reg}$. For ∇ an integrable connection on N, the sheaf of holomorphic horizontal sections

$$N^{\nabla}=\{n\in N^{an}:\nabla_y(n)=0\text{ for all }y\in\mathfrak{h}\}$$

is a W-equivariant local system on $\mathfrak{h}_{reg}$. Hence it identifies with a local system on $\mathfrak{h}_{reg}/W$. So it yields a finite dimensional representation of $\mathbb{C}B(W,\mathfrak{h})$. For $M\in\mathcal{O}_c(W,\mathfrak{h})$ it is proved in [GGOR, Theorem 5.13] that the action of $\mathbb{C}B(W,\mathfrak{h})$ on $(M_{\mathfrak{h}_{reg}})^{\nabla}$ factors through the Hecke algebra $\mathscr{H}_q(W,\mathfrak{h})$. The formula for the parameter q is given in [GGOR, Section 5.2].

The Knizhnik-Zamolodchikov functor is the functor

$$\mathrm{KZ}(W,\mathfrak{h}):\mathcal{O}_c(W,\mathfrak{h})\to\mathscr{H}_q(W,\mathfrak{h})\text{-mod},\quad M\mapsto(M_{\mathfrak{h}_{reg}})^{\nabla}.$$

By definition it is exact. Let us recall some of its properties following [GGOR]. Assume in the rest of this subsection that *the algebras $\mathscr{H}_q(W,\mathfrak{h})$ and $\mathbb{C}W$ have the same dimension over $\mathbb{C}$*. We abbreviate $\mathrm{KZ}=\mathrm{KZ}(W,\mathfrak{h})$. The functor KZ is represented by a projective object P_{KZ} in $\mathcal{O}_c(W,\mathfrak{h})$. More precisely, there is an algebra homomorphism

$$\rho:\mathscr{H}_q(W,\mathfrak{h})\to\mathrm{End}_{\mathcal{O}_c(W,\mathfrak{h})}(P_{\mathrm{KZ}})^{\mathrm{op}}$$

such that KZ is isomorphic to the functor $\mathrm{Hom}_{\mathcal{O}_c(W,\mathfrak{h})}(P_{\mathrm{KZ}},-)$. By [GGOR, Theorem 5.15] the homomorphism ρ is an isomorphism. In particular $\mathrm{KZ}(P_{\mathrm{KZ}})$ is isomorphic to $\mathscr{H}_q(W,\mathfrak{h})$ as $\mathscr{H}_q(W,\mathfrak{h})$-modules.

Now, recall that the center of a category $\mathcal{C}$ is the algebra $Z(\mathcal{C})$ of endomorphisms of the identity functor $Id_{\mathcal{C}}$. So there is a canonical map

$$Z(\mathcal{O}_c(W,\mathfrak{h}))\to\mathrm{End}_{\mathcal{O}_c(W,\mathfrak{h})}(P_{\mathrm{KZ}}).$$

The composition of this map with ρ^{-1} yields an algebra homomorphism

$$\gamma:Z(\mathcal{O}_c(W,\mathfrak{h}))\to Z(\mathscr{H}_q(W,\mathfrak{h})),$$

where $Z(\mathscr{H}_q(W,\mathfrak{h}))$ denotes the center of $\mathscr{H}_q(W,\mathfrak{h})$.

Lemma 1.3. *(1) The homomorphism γ is an isomorphism.*

(2) For a module M in $\mathcal{O}_c(W,\mathfrak{h})$ and an element f in $Z(\mathcal{O}_c(W,\mathfrak{h}))$ the morphism

$$\mathrm{KZ}(f(M)):\mathrm{KZ}(M)\to\mathrm{KZ}(M)$$

is the multiplication by $\gamma(f)$.

See [GGOR, Corollary 5.18] for (1). Part (2) follows from the construction of γ.

The functor KZ is a quotient functor, see [GGOR, Theorem 5.14]. Therefore it has a right adjoint $S:\mathscr{H}_q(W,\mathfrak{h})\to\mathcal{O}_c(W,\mathfrak{h})$ such that the canonical adjunction map $\mathrm{KZ}\circ S\to\mathrm{Id}_{\mathscr{H}_q(W,\mathfrak{h})}$ is an isomorphism of functors. We have the following proposition.

Proposition 1.4. *Let Q be a projective object in $\mathcal{O}_c(W,\mathfrak{h})$.*

(1) For any object $M\in\mathcal{O}_c(W,\mathfrak{h})$, the following morphism of $\mathbb{C}$-vector spaces is an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_c(W,\mathfrak{h})}(M,Q)\xrightarrow{\sim}\mathrm{Hom}_{\mathscr{H}_q(W)}(\mathrm{KZ}(M),\mathrm{KZ}(Q)),\quad f\mapsto\mathrm{KZ}(f).$$

In particular, the functor KZ *is fully faithful over* $\mathrm{Proj}_c(W,\mathfrak{h})$.

(2)The canonical adjunction map gives an isomorphism $Q \overset{\sim}{\to} S\circ \mathrm{KZ}(Q)$.

See [GGOR, Theorems 5.3, 5.16].

1.6. **Parabolic restriction and induction for rational DAHA's.** From now on we will always assume that $\mathfrak{h}^W = 1$. Recall from Section 1.2 that $W' \subset W$ is the stabilizer of a point $b \in \mathfrak{h}$ and that $\overline{\mathfrak{h}} = \mathfrak{h}/\mathfrak{h}^{W'}$. Let us recall from [BE] the definition of the parabolic restriction and induction functors

$$\mathrm{Res}_b : \mathcal{O}_c(W,\mathfrak{h}) \to \mathcal{O}_{c'}(W',\overline{\mathfrak{h}})\,, \quad \mathrm{Ind}_b : \mathcal{O}_{c'}(W',\overline{\mathfrak{h}}) \to \mathcal{O}_c(W,\mathfrak{h}).$$

First we need some notation. For any point $p \in \mathfrak{h}$ we write $\mathbb{C}[[\mathfrak{h}]]_p$ for the completion of $\mathbb{C}[\mathfrak{h}]$ at p, and we write $\widehat{\mathbb{C}[\mathfrak{h}]}_p$ for the completion of $\mathbb{C}[\mathfrak{h}]$ at the W-orbit of p in $\mathfrak{h}$. Note that we have $\mathbb{C}[[\mathfrak{h}]]_0 = \widehat{\mathbb{C}[\mathfrak{h}]}_0$. For any $\mathbb{C}[\mathfrak{h}]$-module M let

$$\widehat{M_p} = \widehat{\mathbb{C}[\mathfrak{h}]}_p \otimes_{\mathbb{C}[\mathfrak{h}]} M.$$

The completions $\widehat{H}_c(W,\mathfrak{h})_b$, $\widehat{H}_{c'}(W',\mathfrak{h})_0$ are well defined algebras. We denote by $\widehat{\mathcal{O}}_c(W,\mathfrak{h})_b$ the category of $\widehat{H}_c(W,\mathfrak{h})_b$-modules that are finitely generated over $\widehat{\mathbb{C}[\mathfrak{h}]}_b$, and we denote by $\widehat{\mathcal{O}}_{c'}(W',\mathfrak{h})_0$ the category of $\widehat{H}_{c'}(W',\mathfrak{h})_0$-modules that are finitely generated over $\widehat{\mathbb{C}[\mathfrak{h}]}_0$. Let $P = \mathrm{Fun}_{W'}(W, \widehat{H}_c(W',\mathfrak{h})_0)$ be the set of W'-invariant maps from W to $\widehat{H}_c(W',\mathfrak{h})_0$. Let $Z(W,W',\widehat{H}_c(W',\mathfrak{h})_0)$ be the ring of endomorphisms of the right $\widehat{H}_c(W',\mathfrak{h})_0$-module P. We have the following proposition given by [BE, Theorem 3.2].

Proposition 1.5. *There is an isomorphism of algebras*

$$\Theta : \widehat{H}_c(W,\mathfrak{h})_b \longrightarrow Z(W,W',\widehat{H}_{c'}(W',\mathfrak{h})_0)$$

defined as follows: for $f\in P$, $\alpha \in \mathfrak{h}^*$, $a\in\mathfrak{h}$, $u \in W$,

$$\begin{aligned}
(\Theta(u)f)(w) &= f(wu),\\
(\Theta(x_\alpha)f)(w) &= (x^{(b)}_{w\alpha} + \alpha(w^{-1}b))f(w),\\
(\Theta(y_a)f)(w) &= y^{(b)}_{wa}f(w) + \sum_{s\in\mathcal{S}, s\notin W'} \frac{2c_s}{1-\lambda_s}\frac{\alpha_s(wa)}{x^{(b)}_{\alpha_s}+\alpha_s(b)}(f(sw)-f(w)),
\end{aligned}$$

where $x_\alpha \in \mathfrak{h}^* \subset H_c(W,\mathfrak{h})$, $x^{(b)}_\alpha \in \mathfrak{h}^* \subset H_{c'}(W',\mathfrak{h})$, $y_a \in \mathfrak{h} \subset H_c(W,\mathfrak{h})$, $y^{(b)}_a \in \mathfrak{h} \subset H_{c'}(W',\mathfrak{h})$.

Using Θ we will identify $\widehat{H}_c(W,\mathfrak{h})_b$-modules with $Z(W,W',\widehat{H}_{c'}(W',\mathfrak{h})_0)$-modules. So the module $P = \mathrm{Fun}_{W'}(W,\widehat{H}_c(W',\mathfrak{h})_0)$ becomes an $(\widehat{H}_c(W,\mathfrak{h})_b, \widehat{H}_{c'}(W',\mathfrak{h})_0)$-bimodule. Hence for any $N \in \widehat{\mathcal{O}}_{c'}(W',\mathfrak{h})_0$ the module $P\otimes_{\widehat{H}_{c'}(W',\mathfrak{h})_0} N$ lives in $\widehat{\mathcal{O}}_c(W,\mathfrak{h})_b$. It is naturally identified with $\mathrm{Fun}_{W'}(W,N)$, the set of W'-invariant maps from W to N. For any $\mathbb{C}[\mathfrak{h}^*]$-module M write $E(M) \subset M$ for the locally nilpotent part of M under the action of $\mathfrak{h}$.

The ingredients for defining the functors Res_b and Ind_b consist of:

- the adjoint pair of functors $(\widehat{\ }_b, E^b)$ with

$$\widehat{\ }_b : \mathcal{O}_c(W,\mathfrak{h}) \to \widehat{\mathcal{O}}_c(W,\mathfrak{h})_b, \quad M \mapsto \widehat{M}_b,$$

$$E^b : \widehat{\mathcal{O}}_c(W,\mathfrak{h})_b \to \mathcal{O}_c(W,\mathfrak{h}), \quad N \to E(N),$$

- the Morita equivalence

$$J : \widehat{\mathcal{O}}_{c'}(W',\mathfrak{h})_0 \to \widehat{\mathcal{O}}_c(W,\mathfrak{h})_b, \quad N \mapsto \mathrm{Fun}_{W'}(W,N),$$

  and its quasi-inverse R given in Section 1.7 below,

- the equivalence of categories

$$E:\widehat{\mathcal{O}}_{c'}(W',\mathfrak{h})_0\to\mathcal{O}_{c'}(W',\mathfrak{h}),\quad M\mapsto E(M)$$

 and its quasi-inverse given by $N\mapsto\widehat{N}_0$,
- the equivalence of categories

$$\zeta:\mathcal{O}_{c'}(W',\mathfrak{h})\to\mathcal{O}_{c'}(W',\overline{\mathfrak{h}}),\quad M\mapsto\{v\in M:\ yv=0,\ \text{ for all } y\in\mathfrak{h}^{W'}\}\tag{1.10}$$

 and its quasi-inverse ζ^{-1} given in Section 1.8 below.

For $M\in\mathcal{O}_c(W,\mathfrak{h})$ and $N\in\mathcal{O}_{c'}(W',\overline{\mathfrak{h}})$ the functors Res_b and Ind_b are defined by

$$\mathrm{Res}_b(M)=\zeta\circ E\circ R(\widehat{M}_b),\tag{1.11}$$
$$\mathrm{Ind}_b(N)=E^b\circ J(\widehat{\zeta^{-1}(N)}_0).$$

We refer to [BE, Section 2,3] for details.

1.7. **The idempotent x_{pr} and the functor R.** We give some details on the isomorphism Θ for a future use. Fix elements $1=u_1,u_2,\ldots,u_r$ in W such that $W=\bigsqcup_{i=1}^r W'u_i$. Let $\mathrm{Mat}_r(\widehat{H}_{c'}(W',\mathfrak{h})_0)$ be the algebra of $r\times r$ matrices with coefficients in $\widehat{H}_{c'}(W',\mathfrak{h})_0$. We have an algebra isomorphism

$$\begin{aligned}\Phi:Z(W,W',\widehat{H}_{c'}(W',\mathfrak{h})_0)&\to\mathrm{Mat}_r(\widehat{H}_{c'}(W',\mathfrak{h})_0),\\ A&\mapsto(\Phi(A)_{ij})_{1\leqslant i,j\leqslant r}\end{aligned}\tag{1.12}$$

such that

$$(Af)(u_i)=\sum_{j=1}^r\Phi(A)_{ij}f(u_j),\qquad\text{for all } f\in P,\ 1\leqslant i\leqslant r.$$

Denote by E_{ij}, $1\leqslant i,j\leqslant r$, the elementary matrix in $\mathrm{Mat}_r(\widehat{H}_{c'}(W',\mathfrak{h})_0)$ with coefficient 1 in the position (i,j) and zero elsewhere. Note that the algebra isomorphism

$$\Phi\circ\Theta:\widehat{H}_c(W,\mathfrak{h})_b\xrightarrow{\sim}\mathrm{Mat}_r(\widehat{H}_{c'}(W',\mathfrak{h})_0)$$

restricts to an isomorphism of subalgebras

$$\widehat{\mathbb{C}[\mathfrak{h}]}_b\cong\bigoplus_{i=1}^r\mathbb{C}[[\mathfrak{h}]]_0E_{ii}.\tag{1.13}$$

Indeed, there is an unique isomorphism of algebras

$$\varpi:\widehat{\mathbb{C}[\mathfrak{h}]}_b\cong\bigoplus_{i=1}^r\mathbb{C}[[\mathfrak{h}]]_{u_i^{-1}b}.\tag{1.14}$$

extending the algebra homomorphism

$$\mathbb{C}[\mathfrak{h}]\to\bigoplus_{i=1}^r\mathbb{C}[\mathfrak{h}],\quad x\mapsto(x,x,\ldots,x),\quad\forall\ x\in\mathfrak{h}^*.$$

For each i consider the isomorphism of algebras

$$\phi_i:\mathbb{C}[[\mathfrak{h}]]_{u_i^{-1}b}\to\mathbb{C}[[\mathfrak{h}]]_0,\quad x\mapsto u_ix+x(u_i^{-1}b),\quad\forall\ x\in\mathfrak{h}^*.$$

The isomorphism (1.13) is exactly the composition of ϖ with the direct sum $\oplus_{i=1}^r\phi_i$. Here E_{ii} is the image of the idempotent in $\widehat{\mathbb{C}[\mathfrak{h}]}_b$ corresponding to the component $\mathbb{C}[[\mathfrak{h}]]_{u_i^{-1}b}$. We will denote by x_{pr} the idempotent in $\widehat{\mathbb{C}[\mathfrak{h}]}_b$ corresponding to $\mathbb{C}[[\mathfrak{h}]]_b$, i.e., $\Phi\circ\Theta(x_{\mathrm{pr}})=E_{11}$. Then the following functor

$$R:\widehat{\mathcal{O}}_c(W,\mathfrak{h})_b\to\widehat{\mathcal{O}}_{c'}(W',\mathfrak{h})_0,\quad M\mapsto x_{\mathrm{pr}}M$$

is a quasi-inverse of J. Here, the action of $\widehat{H}_{c'}(W',\mathfrak{h})_0$ on $R(M)=x_{\mathrm{pr}}M$ is given by the following formulas deduced from Proposition 1.5. For any $\alpha\in\mathfrak{h}^*$, $w\in W'$, $a\in\mathfrak{h}^*$, $m\in M$ we have

$$x^{(b)}_\alpha x_{\mathrm{pr}}(m) \;=\; x_{\mathrm{pr}}((x_\alpha-\alpha(b))m), \tag{1.15}$$

$$w x_{\mathrm{pr}}(m) \;=\; x_{\mathrm{pr}}(wm), \tag{1.16}$$

$$y^{(b)}_a x_{\mathrm{pr}}(m) \;=\; x_{\mathrm{pr}}((y_a+\sum_{s\in\mathcal{S},\, s\notin W'}\frac{2c_s}{1-\lambda_s}\frac{\alpha_s(a)}{x_{\alpha_s}})m). \tag{1.17}$$

In particular, we have

$$R(M)=\phi_1^*(x_{\mathrm{pr}}(M)) \tag{1.18}$$

as $\mathbb{C}[[\mathfrak{h}]]_0\rtimes W'$-modules. Finally, note that the following equality holds in $\widehat{H}_c(W,\mathfrak{h})_b$

$$x_{\mathrm{pr}}ux_{\mathrm{pr}}=0,\quad \forall\, u\in W-W'. \tag{1.19}$$

1.8. **A quasi-inverse of ζ.** Let us recall from [BE, Section 2.3] the following facts. Let $\mathfrak{h}^{*W'}$ be the subspace of $\mathfrak{h}^*$ consisting of fixed points of W'. Set

$$(\mathfrak{h}^{*W'})^\perp=\{v\in\mathfrak{h}:f(v)=0 \text{ for all } f\in\mathfrak{h}^{*W'}\}.$$

We have a W'-invariant decomposition

$$\mathfrak{h}=(\mathfrak{h}^{*W'})^\perp\oplus\mathfrak{h}^{W'}.$$

The W'-space $(\mathfrak{h}^{*W'})^\perp$ is canonically identified with $\overline{\mathfrak{h}}$. Since the action of W' on $\mathfrak{h}^{W'}$ is trivial, we have an obvious algebra isomorphism

$$H_{c'}(W',\mathfrak{h})\cong H_{c'}(W',\overline{\mathfrak{h}})\otimes\mathcal{D}(\mathfrak{h}^{W'}). \tag{1.20}$$

It maps an element y in the subset $\mathfrak{h}^{W'}$ of $H_{c'}(W',\mathfrak{h})$ to the operator ∂_y in $\mathcal{D}(\mathfrak{h}^{W'})$. Write $\mathcal{O}(1,\mathfrak{h}^{W'})$ for the category of finitely generated $\mathcal{D}(\mathfrak{h}^{W'})$-modules that are ∂_y-locally nilpotent for all $y\in\mathfrak{h}^{W'}$. The algebra isomorphism above yields an equivalence of categories

$$\mathcal{O}_{c'}(W',\mathfrak{h})\cong\mathcal{O}_{c'}(W',\overline{\mathfrak{h}})\otimes\mathcal{O}(1,\mathfrak{h}^{W'}).$$

The functor ζ in (1.10) is an equivalence, because it is induced by the functor

$$\mathcal{O}(1,\mathfrak{h}^{W'})\xrightarrow{\sim}\mathbb{C}\text{-mod},\quad M\to\{m\in M,\partial_y(m)=0 \text{ for all } y\in\mathfrak{h}^{W'}\},$$

which is an equivalence by Kashiwara's lemma upon taking Fourier transforms. In particular, a quasi-inverse of ζ is given by

$$\zeta^{-1}:\mathcal{O}_{c'}(W',\overline{\mathfrak{h}})\to\mathcal{O}_{c'}(W',\mathfrak{h}),\quad N\mapsto N\otimes\mathbb{C}[\mathfrak{h}^{W'}], \tag{1.21}$$

where $\mathbb{C}[\mathfrak{h}^{W'}]\in\mathcal{O}(1,\mathfrak{h}^{W'})$ is the polynomial representation of $\mathcal{D}(\mathfrak{h}^{W'})$.

Moreover, the functor ζ maps a standard module in $\mathcal{O}_{c'}(W',\mathfrak{h})$ to a standard module in $\mathcal{O}_{c'}(W',\overline{\mathfrak{h}})$. Indeed, for any $\xi\in\mathrm{Irr}(W')$, we have an isomorphism of $H_{c'}(W',\mathfrak{h})$-modules

$$H_{c'}(W',\mathfrak{h})\otimes_{\mathbb{C}[\mathfrak{h}^*]\rtimes W'}\xi=(H_{c'}(W',\overline{\mathfrak{h}})\otimes_{\mathbb{C}[(\overline{\mathfrak{h}})^*]\rtimes W'}\xi)\otimes(\mathcal{D}(\mathfrak{h}^{W'})\otimes_{\mathbb{C}[(\mathfrak{h}^{W'})^*]}\mathbb{C}).$$

On the right hand side $\mathbb{C}$ denotes the trivial module of $\mathbb{C}[(\mathfrak{h}^{W'})^*]$, and the latter is identified with the subalgebra of $\mathcal{D}(\mathfrak{h}^{W'})$ generated by ∂_y for all $y\in\mathfrak{h}^{W'}$. We have

$$\mathcal{D}(\mathfrak{h}^{W'})\otimes_{\mathbb{C}[(\mathfrak{h}^{W'})^*]}\mathbb{C}\cong\mathbb{C}[\mathfrak{h}^{W'}]$$

as $\mathcal{D}(\mathfrak{h}^{W'})$-modules. So ζ maps the standard module $\Delta(\xi)$ for $H_{c'}(W',\mathfrak{h})$ to the standard module $\Delta(\xi)$ for $H_{c'}(W',\overline{\mathfrak{h}})$.

1.9. Here are some properties of Res_b and Ind_b.

Proposition 1.6. (1) *Both functors* Res_b *and* Ind_b *are exact. The functor* Res_b *is left adjoint to* Ind_b. *In particular the functor* Res_b *preserves projective objects and* Ind_b *preserves injective objects.*

(2) *Let* $\mathrm{Res}^W_{W'}$ *and* $\mathrm{Ind}^W_{W'}$ *be respectively the restriction and induction functors of groups. We have the following commutative diagram*

$$\begin{tikzcd} K(\mathcal{O}_c(W,\mathfrak{h})) \arrow[r, "\omega", "\sim"'] \arrow[d, shift left, "\mathrm{Res}_b"] & K(\mathbb{C}W) \arrow[d, shift left, "\mathrm{Res}^W_{W'}"] \\ K(\mathcal{O}_{c'}(W',\overline{\mathfrak{h}})) \arrow[r, "\omega'", "\sim"'] \arrow[u, shift left, "\mathrm{Ind}_b"] & K(\mathbb{C}W'). \arrow[u, shift left, "\mathrm{Ind}^W_{W'}"] \end{tikzcd}$$

Here the isomorphism ω *(resp.* ω'*) is given by mapping* $[\Delta(\xi)]$ *to* $[\xi]$ *for any* $\xi \in \mathrm{Irr}(W)$ *(resp.* $\xi \in \mathrm{Irr}(W')$*).*

See [BE, Proposition 3.9, Theorem 3.10] for (1), [BE, Proposition 3.14] for (2).

1.10. **Restriction of modules having a standard filtration.** In the rest of Section 1, we study the actions of the restriction functors on modules having a standard filtration in $\mathcal{O}_c(W,\mathfrak{h})$ (Proposition 1.9). We will need the following lemmas.

Lemma 1.7. *Let* M *be a module in* $\mathcal{O}^{\Delta}_c(W,\mathfrak{h})$.

(1) There is a finite dimensional subspace V *of* M *such that* V *is stable under the action of* $\mathbb{C}W$ *and the map*

$$\mathbb{C}[\mathfrak{h}]\otimes V \to M, \quad p\otimes v\mapsto pv$$

is an isomorphism of $\mathbb{C}[\mathfrak{h}]\rtimes W$*-modules.*

(2) The map $\omega : K(\mathcal{O}_c(W,\mathfrak{h}))\to K(\mathbb{C}W)$ *in Proposition 1.6(2) satisfies*

$$\omega([M]) = [V]. \tag{1.22}$$

Proof. Let

$$0 = M_0 \subset M_1 \subset \ldots \subset M_l = M$$

be a filtration of M such that for any $1 \leqslant i \leqslant l$ we have $M_i/M_{i-1}\cong \Delta(\xi_i)$ for some $\xi_i \in \mathrm{Irr}(W)$. We prove (1) and (2) by recurrence on l. If $l = 1$, then M is a standard module. Both (1) and (2) hold by definition. For $l > 1$, by induction we may suppose that there is a subspace V' of M_{l-1} such that the properties in (1) and (2) are satisfied for M_{l-1} and V'. Now, consider the exact sequence

$$0 \longrightarrow M_{l-1} \longrightarrow M \xrightarrow{j} \Delta(\xi_l) \longrightarrow 0$$

From the isomorphism of $\mathbb{C}[\mathfrak{h}]\rtimes W$-modules $\Delta(\xi_l)\cong \mathbb{C}[\mathfrak{h}]\otimes\xi$ we see that $\Delta(\xi_l)$ is a projective $\mathbb{C}[\mathfrak{h}]\rtimes W$-module. Hence there exists a morphism of $\mathbb{C}[\mathfrak{h}]\rtimes W$-modules $s : \Delta(\xi_l)\to M$ that provides a section of j. Let $V = V'\oplus s(\xi_l)\subset M$. It is stable under the action of $\mathbb{C}W$. The map $\mathbb{C}[\mathfrak{h}]\otimes V\to M$ in (1) is an injective morphism of $\mathbb{C}[\mathfrak{h}]\rtimes W$-modules. Its image is $M_{l-1}\oplus s(\Delta(\xi))$, which is equal to M. So it is an isomorphism. We have

$$\omega([M]) = \omega([M_{l-1}]) + \omega([\Delta(\xi_l)]),$$

by assumption $\omega([M_{l-1}]) = [V']$, so $\omega([M]) = [V'] + [\xi_l] = [V]$. □

Lemma 1.8. *(1) Let* M *be a* $\widehat{H}_c(W,\mathfrak{h})_0$*-module free over* $\mathbb{C}[[\mathfrak{h}]]_0$*. If there exist generalized eigenvectors* $v_1,\ldots v_n$ *of* **eu** *which form a basis of* M *over* $\mathbb{C}[[\mathfrak{h}]]_0$*, then for* $f_1,\ldots,f_n\in\mathbb{C}[[\mathfrak{h}]]_0$ *the element* $m=\sum_{i=1}^n f_iv_i$ *is* **eu**-*finite if and only if* $f_1,\ldots,f_n$ *all belong to* $\mathbb{C}[\mathfrak{h}]$.

(2) Let N be an object in $\mathcal{O}_c(W,\mathfrak{h})$. If $\widehat{N}_0$ is a free $\mathbb{C}[[\mathfrak{h}]]_0$-module, then N is a free $\mathbb{C}[\mathfrak{h}]$-module. It admits a basis consisting of generalized eigenvectors $v_1,\dots,v_n$ of **eu**.

Proof. (1) It follows from the proof of [BE, Theorem 2.3].

(2) Since N belongs to $\mathcal{O}_c(W,\mathfrak{h})$, it is finitely generated over $\mathbb{C}[\mathfrak{h}]$. Denote by $\mathfrak{m}$ the maximal ideal of $\mathbb{C}[[\mathfrak{h}]]_0$. The canonical map $N\to\widehat{N}_0/\mathfrak{m}\widehat{N}_0$ is surjective. So there exist $v_1,\dots,v_n$ in N such that their images form a basis of $\widehat{N}_0/\mathfrak{m}\widehat{N}_0$ over $\mathbb{C}$. Moreover, we may choose $v_1,\dots,v_n$ to be generalized eigenvectors of **eu**, because the **eu**-action on N is locally finite. Since $\widehat{N}_0$ is free over $\mathbb{C}[[\mathfrak{h}]]_0$, Nakayama's lemma yields that $v_1,\dots,v_n$ form a basis of $\widehat{N}_0$ over $\mathbb{C}[[\mathfrak{h}]]_0$. By part (1) the set N' of **eu**-finite elements in $\widehat{N}_0$ is the free $\mathbb{C}[\mathfrak{h}]$-submodule generated by $v_1,\dots,v_n$. On the other hand, since $\widehat{N}_0$ belongs to $\widehat{\mathcal{O}}_c(W,\mathfrak{h})_0$, by [BE, Proposition 2.4] an element in $\widehat{N}_0$ is $\mathfrak{h}$-nilpotent if and only if it is **eu**-finite. So $N'=E(\widehat{N}_0)$. On the other hand, the canonical inclusion $N\subset E(\widehat{N}_0)$ is an equality by [BE, Theorem 3.2]. Hence $N=N'$. This implies that N is free over $\mathbb{C}[\mathfrak{h}]$, with a basis given by $v_1,\dots,v_n$, which are generalized eigenvectors of **eu**. $\square$

Proposition 1.9. *Let M be an object in $\mathcal{O}_c^{\Delta}(W,\mathfrak{h})$.*

(1) The object $\mathrm{Res}_b(M)$ has a standard filtration.

(2) Let V be a subspace of M that has the properties of Lemma 1.7(1). Then there is an isomorphism of $\mathbb{C}[\overline{\mathfrak{h}}]\rtimes W'$-modules

$$\mathrm{Res}_b(M)\cong\mathbb{C}[\overline{\mathfrak{h}}]\otimes\mathrm{Res}^W_{W'}(V).$$

Proof. (1) By the end of Section 1.8 the equivalence ζ maps a standard module in $\mathcal{O}_{c'}(W',\mathfrak{h})$ to a standard one in $\mathcal{O}_{c'}(W',\overline{\mathfrak{h}})$. Hence to prove that $\mathrm{Res}_b(M)=\zeta\circ E\circ R(\widehat{M_b})$ has a standard filtration, it is enough to show that $N=E\circ R(\widehat{M_b})$ has one. We claim that the module N is free over $\mathbb{C}[\mathfrak{h}]$. So the result follows from Lemma 1.1(2).

Let us prove the claim. Recall from (1.18) that we have $R(\widehat{M_b})=\phi_1^*(x_{\mathrm{pr}}\widehat{M_b})$ as $\mathbb{C}[[\mathfrak{h}]]_0\rtimes W'$-modules. Using the isomorphism of $\mathbb{C}[\mathfrak{h}]\rtimes W$-modules $M\cong\mathbb{C}[\mathfrak{h}]\otimes V$ given in Lemma 1.7(1), we deduce an isomorphism of $\mathbb{C}[[\mathfrak{h}]]_0\rtimes W'$-modules

$$\begin{aligned} R(\widehat{M_b}) &\cong \phi_1^*(x_{\mathrm{pr}}(\widehat{\mathbb{C}[\mathfrak{h}]}_b\otimes V))\\ &\cong \mathbb{C}[[\mathfrak{h}]]_0\otimes V. \end{aligned}$$

So the module $R(\widehat{M_b})$ is free over $\mathbb{C}[[\mathfrak{h}]]_0$. The completion of the module N at 0 is isomorphic to $R(\widehat{M_b})$. By Lemma 1.8(2) the module N is free over $\mathbb{C}[\mathfrak{h}]$. The claim is proved.

(2) Since $\mathrm{Res}_b(M)$ has a standard filtration, by Lemma 1.7 there exists a finite dimensional vector space $V'\subset\mathrm{Res}_b(M)$ such that V' is stable under the action of $\mathbb{C}W'$ and we have an isomorphism of $\mathbb{C}[\overline{\mathfrak{h}}]\rtimes W'$-modules

$$\mathrm{Res}_b(M)\cong\mathbb{C}[\overline{\mathfrak{h}}]\otimes V'.$$

Moreover, we have $\omega'([\mathrm{Res}_b(M)])=[V']$ where ω' is the map in Proposition 1.6(2). The same proposition yields that $\mathrm{Res}^W_{W'}(\omega[M])=\omega'([\mathrm{Res}_b(M)])$. Since $\omega([M])=[V]$ by (1.22), the $\mathbb{C}W'$-module V' is isomorphic to $\mathrm{Res}^W_{W'}(V)$. So we have an isomorphism of $\mathbb{C}[\overline{\mathfrak{h}}]\rtimes W'$-modules

$$\mathrm{Res}_b(M)\cong\mathbb{C}[\overline{\mathfrak{h}}]\otimes\mathrm{Res}^W_{W'}(V).$$

$\square$

2. KZ commutes with restriction functors

In this section, we relate the restriction and induction functors for rational DAHA's to the corresponding functors for Hecke algebras via the functor KZ. We will always assume that the Hecke algebras have the same dimension as the corresponding group algebras. Thus the Knizhnik-Zamolodchikov functors admit the properties recalled in Section 1.5.

2.1. Let W be a complex reflection group acting on $\mathfrak{h}$. Let b be a point in $\mathfrak{h}$ and let W' be its stabilizer in W. We will abbreviate $\mathrm{KZ}=\mathrm{KZ}(W,\mathfrak{h})$, $\mathrm{KZ}'=\mathrm{KZ}(W',\overline{\mathfrak{h}})$.

Theorem 2.1. *There is an isomorphism of functors*

$$\mathrm{KZ}'\circ\mathrm{Res}_b\cong {}^{\mathscr{H}}\mathrm{Res}_{W'}^{W}\circ\mathrm{KZ}\,.$$

Proof. We will regard $\mathrm{KZ}:\mathcal{O}_c(W,\mathfrak{h})\to\mathscr{H}_q(W)$-mod as a functor from $\mathcal{O}_c(W,\mathfrak{h})$ to B_W-mod in the obvious way. Similarly we will regard KZ' as a functor to $B_{W'}$-mod. Recall the inclusion $\imath:B_{W'}\hookrightarrow B_W$ from (1.6). The theorem amounts to prove that for any $M\in\mathcal{O}_c(W,\mathfrak{h})$ there is a natural isomorphism of $B_{W'}$-modules

$$\mathrm{KZ}'\circ\mathrm{Res}_b(M)\cong\imath_*\circ\mathrm{KZ}(M). \tag{2.1}$$

Step 1. Recall the functor $\zeta:\mathcal{O}_{c'}(W',\mathfrak{h})\to\mathcal{O}_{c'}(W',\overline{\mathfrak{h}})$ from (1.10) and its quasi-inverse ζ^{-1} in (1.21). Let

$$N=\zeta^{-1}(\mathrm{Res}_b(M)).$$

We have $N\cong\mathrm{Res}_b(M)\otimes\mathbb{C}[\mathfrak{h}^{W'}]$. Since the canonical epimorphism $\mathfrak{h}\to\overline{\mathfrak{h}}$ induces a fibration $\mathfrak{h}'_{reg}\to\overline{\mathfrak{h}}_{reg}$, see Section 1.2, we have

$$N_{\mathfrak{h}'_{reg}}\cong\mathrm{Res}_b(M)_{\overline{\mathfrak{h}}_{reg}}\otimes\mathbb{C}[\mathfrak{h}^{W'}]. \tag{2.2}$$

By Dunkl isomorphisms, the left hand side is a $\mathcal{D}(\mathfrak{h}'_{reg})\rtimes W'$-module while the right hand side is a $(\mathcal{D}(\overline{\mathfrak{h}}_{reg})\rtimes W')\otimes\mathcal{D}(\mathfrak{h}^{W'})$-module. Identify these two algebras in the obvious way. The isomorphism (2.2) is compatible with the W'-equivariant $\mathcal{D}$-module structures. Hence we have

$$(N_{\mathfrak{h}'_{reg}})^{\nabla}\cong(\mathrm{Res}_b(M)_{\overline{\mathfrak{h}}_{reg}})^{\nabla}\otimes\mathbb{C}[\mathfrak{h}^{W'}]^{\nabla}.$$

Since $\mathbb{C}[\mathfrak{h}^{W'}]^{\nabla}=\mathbb{C}$, this yields a natural isomorphism

$$\ell_*\circ\mathrm{KZ}(W',\mathfrak{h})(N)\cong\mathrm{KZ}'\circ\mathrm{Res}_b(M),$$

where ℓ is the homomorphism defined in (1.2).

Step 2. Consider the W'-equivariant algebra isomorphism

$$\phi:\mathbb{C}[\mathfrak{h}]\to\mathbb{C}[\mathfrak{h}],\quad x\mapsto x+x(b)\text{ for }x\in\mathfrak{h}^*.$$

It induces an isomorphism $\hat{\phi}:\mathbb{C}[[\mathfrak{h}]]_b\xrightarrow{\sim}\mathbb{C}[[\mathfrak{h}]]_0$. The latter yields an algebra isomorphism

$$\mathbb{C}[[\mathfrak{h}]]_b\otimes_{\mathbb{C}[\mathfrak{h}]}\mathbb{C}[\mathfrak{h}_{reg}]\simeq\mathbb{C}[[\mathfrak{h}]]_0\otimes_{\mathbb{C}[\mathfrak{h}]}\mathbb{C}[\mathfrak{h}'_{reg}].$$

To see this note first that by definition, the left hand side is $\mathbb{C}[[\mathfrak{h}]]_b[\alpha_s^{-1},s\in\mathcal{S}]$. For $s\in\mathcal{S}$, $s\notin W'$ the element α_s is invertible in $\mathbb{C}[[\mathfrak{h}]]_b$, so we have

$$\mathbb{C}[[\mathfrak{h}]]_b\otimes_{\mathbb{C}[\mathfrak{h}]}\mathbb{C}[\mathfrak{h}_{reg}]=\mathbb{C}[[\mathfrak{h}]]_b[\alpha_s^{-1},s\in\mathcal{S}\cap W'].$$

For $s\in\mathcal{S}\cap W'$ we have $\alpha_s(b)=0$, so $\hat{\phi}(\alpha_s)=\alpha_s$. Hence

$$\begin{aligned}\hat{\phi}(\mathbb{C}[[\mathfrak{h}]]_b)[\hat{\phi}(\alpha_s)^{-1},s\in\mathcal{S}\cap W'] &= \mathbb{C}[[\mathfrak{h}]]_0[\alpha_s^{-1},s\in\mathcal{S}\cap W']\\ &= \mathbb{C}[[\mathfrak{h}]]_0\otimes_{\mathbb{C}[\mathfrak{h}]}\mathbb{C}[\mathfrak{h}'_{reg}].\end{aligned}$$

Step 3. We will assume in Steps 3, 4, 5 that M is a module in $\mathcal{O}_c^\Delta(W,\mathfrak{h})$. In this step we prove that N is isomorphic to $\phi^*(M)$ as $\mathbb{C}[\mathfrak{h}]\rtimes W'$-modules. Let V be a subspace of M as in Lemma 1.7(1). So we have an isomorphism of $\mathbb{C}[\mathfrak{h}]\rtimes W$-modules

$$M\cong\mathbb{C}[\mathfrak{h}]\otimes V. \tag{2.3}$$

Also, by Proposition 1.9(2) there is an isomorphism of $\mathbb{C}[\mathfrak{h}]\rtimes W'$-modules

$$N\quad\cong\quad\mathbb{C}[\mathfrak{h}]\otimes\mathrm{Res}^W_{W'}(V).$$

So N is isomorphic to $\phi^*(M)$ as $\mathbb{C}[\mathfrak{h}]\rtimes W'$-modules.

Step 4. In this step we compare $((\widehat{\phi^*(M)})_0)_{\mathfrak{h}'_{reg}}$ and $(\widehat{N}_0)_{\mathfrak{h}'_{reg}}$ as $\widehat{\mathcal{D}(\mathfrak{h}'_{reg})}_0$-modules. The definition of these $\widehat{\mathcal{D}(\mathfrak{h}'_{reg})}_0$-module structures will be given below in terms of connections. By (1.11) we have $N=E\circ R(\widehat{M}_b)$, so we have $\widehat{N}_0\cong R(\widehat{M}_b)$. Next, by (1.18) we have an isomorphism of $\mathbb{C}[[\mathfrak{h}]]_0\rtimes W'$-modules

$$\begin{aligned} R(\widehat{M}_b)&=\hat{\phi}^*(x_{\mathrm{pr}}(\widehat{M}_b))\\ &=(\widehat{\phi^*(M)})_0.\end{aligned}$$

So we get an isomorphism of $\mathbb{C}[[\mathfrak{h}]]_0\rtimes W'$-modules

$$\hat{\Psi}:(\widehat{\phi^*(M)})_0\rightarrow\widehat{N}_0.$$

Now, let us consider connections on these modules. Note that by Step 2 we have

$$((\widehat{\phi^*(M)})_0)_{\mathfrak{h}'_{reg}}=\hat{\phi}^*(x_{\mathrm{pr}}(\widehat{M}_b)_{\mathfrak{h}_{reg}}).$$

Write ∇ for the connection on $M_{\mathfrak{h}_{reg}}$ given by the Dunkl isomorphism for $H_c(W,\mathfrak{h}_{reg})$. We equip $((\widehat{\phi^*(M)})_0)_{\mathfrak{h}'_{reg}}$ with the connection $\tilde{\nabla}$ given by

$$\tilde{\nabla}_a(x_{\mathrm{pr}}m)=x_{\mathrm{pr}}(\nabla_a(m)),\quad\forall\ m\in(\widehat{M}_b)_{\mathfrak{h}_{reg}},\ a\in\mathfrak{h}.$$

Let $\nabla^{(b)}$ be the connection on $N_{\mathfrak{h}'_{reg}}$ given by the Dunkl isomorphism for $H_{c'}(W',\mathfrak{h}'_{reg})$. This restricts to a connection on $(\widehat{N}_0)_{\mathfrak{h}'_{reg}}$. We claim that Ψ is compatible with these connections, i.e., we have

$$\nabla^{(b)}_a(x_{\mathrm{pr}}m)=x_{\mathrm{pr}}\nabla_a(m),\quad\forall\ m\in(\widehat{M}_b)_{\mathfrak{h}_{reg}}. \tag{2.4}$$

Recall the subspace V of M from Step 3. By Lemma 1.7(1) the map

$$(\widehat{\mathbb{C}[\mathfrak{h}]}_b\otimes_{\mathbb{C}[\mathfrak{h}]}\mathbb{C}[\mathfrak{h}_{reg}])\otimes V\rightarrow(\widehat{M}_b)_{\mathfrak{h}_{reg}},\quad p\otimes v\mapsto pv$$

is a bijection. So it is enough to prove (2.4) for $m=pv$ with $p\in\widehat{\mathbb{C}[\mathfrak{h}]}_b\otimes_{\mathbb{C}[\mathfrak{h}]}\mathbb{C}[\mathfrak{h}_{reg}]$, $v\in V$. We have

$$\begin{aligned}\nabla^{(b)}_a(x_{\mathrm{pr}}pv)&=(y^{(b)}_a-\sum_{s\in\mathcal{S}\cap W'}\frac{2c_s}{1-\lambda_s}\frac{\alpha_s(a)}{x^{(b)}_{\alpha_s}}(s-1))(x_{\mathrm{pr}}pv)\\ &=x_{\mathrm{pr}}(y_a+\sum_{s\in\mathcal{S},s\notin W'}\frac{2c_s}{1-\lambda_s}\frac{\alpha_s(a)}{x_{\alpha_s}}-\\ &\quad-\sum_{s\in\mathcal{S}\cap W'}\frac{2c_s}{1-\lambda_s}\frac{\alpha_s(a)}{x_{\alpha_s}}(s-1))(x_{\mathrm{pr}}pv)\\ &=x_{\mathrm{pr}}(\nabla_a+\sum_{s\in\mathcal{S},s\notin W'}\frac{2c_s}{1-\lambda_s}\frac{\alpha_s(a)}{x_{\alpha_s}}s)(x_{\mathrm{pr}}pv)\\ &=x_{\mathrm{pr}}\nabla_a(x_{\mathrm{pr}}pv).\end{aligned}\tag{2.5}$$

Here the first equality is by the Dunkl isomorphism for $H_{c'}(W',\mathfrak{h}'_{reg})$. The second is by (1.15), (1.16), (1.17) and the fact that $x^2_{\mathrm{pr}}=x_{\mathrm{pr}}$. The third is by the Dunkl

isomorphism for $H_c(W,\mathfrak{h}_{reg})$. The last is by (1.19). Next, since x_{pr} is the idempotent in $\widehat{\mathbb{C}[\mathfrak{h}]}_b$ corresponding to the component $\mathbb{C}[[\mathfrak{h}]]_b$ in the decomposition (1.14), we have

$$\begin{aligned}\nabla_a(x_{\mathrm{pr}}pv) &= (\partial_a(x_{\mathrm{pr}}p))v + x_{\mathrm{pr}}p\,(\nabla_a v)\\ &= x_{\mathrm{pr}}(\partial_a(p))v + x_{\mathrm{pr}}p\,(\nabla_a v)\\ &= x_{\mathrm{pr}}\nabla_a(pv).\end{aligned}$$

Together with (2.5) this implies that

$$\nabla_a^{(b)}(x_{\mathrm{pr}}pv) = x_{\mathrm{pr}}\nabla_a(pv).$$

So (2.4) is proved.

Step 5. In this step we prove isomorphism (2.1) for $M \in \mathcal{O}_c^{\Delta}(W,\mathfrak{h})$. Here we need some more notation. For $X = \mathfrak{h}$ or $\mathfrak{h}'_{reg}$, let U be an open analytic subvariety of X, write $i : U \hookrightarrow X$ for the canonical embedding. For F an analytic coherent sheaf on X we write $i^*(F)$ for the restriction of F to U. If U contains 0, for an analytic locally free sheaf E over U, we write $\widehat{E}$ for the restriction of E to the formal disc at 0.

Let $\Omega \subset \mathfrak{h}$ be the open ball defined in (1.3). Let $f : \mathfrak{h} \to \mathfrak{h}$ be the morphism defined by ϕ. It maps Ω to an open ball Ω_0 centered at 0. We have

$$f(\Omega \cap \mathfrak{h}_{reg}) = \Omega_0 \cap \mathfrak{h}'_{reg}.$$

Let $u : \Omega_0 \cap \mathfrak{h}'_{reg} \hookrightarrow \mathfrak{h}$ and $v : \Omega \cap \mathfrak{h}_{reg} \hookrightarrow \mathfrak{h}$ be the canonical embeddings. By Step 3 there is an isomorphism of W'-equivariant analytic locally free sheaves over $\Omega_0 \cap \mathfrak{h}'_{reg}$

$$u^*(N^{an}) \cong \phi^*(v^*(M^{an})).$$

By Step 4 there is an isomorphism

$$\widehat{u^*(N^{an})} \xrightarrow{\sim} \phi^*(\widehat{v^*(M^{an})})$$

which is compatible with their connections. It follows from Lemma 2.2 below that there is an isomorphism

$$(u^*(N^{an}))^{\nabla^{(b)}} \cong \phi^*((v^*(M^{an}))^{\nabla}).$$

Since $\Omega_0\cap\mathfrak{h}'_{reg}$ is homotopy equivalent to $\mathfrak{h}'_{reg}$ via u, the left hand side is isomorphic to $(N_{\mathfrak{h}'_{reg}})^{\nabla^{(b)}}$. So we have

$$\kappa_* \circ \jmath_* \circ \mathrm{KZ}(M) \cong \mathrm{KZ}(W',\mathfrak{h})(N),$$

where κ, $\jmath$ are as in (1.4), (1.5). Combined with Step 1 we have the following isomorphisms

$$\begin{aligned}\mathrm{KZ}' \circ \mathrm{Res}_b(M) &\cong \ell_* \circ \mathrm{KZ}(W',\mathfrak{h})(N)\\ &\cong \ell_* \circ \kappa_* \circ \jmath_* \circ \mathrm{KZ}(M)\\ &= \imath_* \circ \mathrm{KZ}(M).\end{aligned} \tag{2.6}$$

They are functorial on M.

Lemma 2.2. *Let E be an analytic locally free sheaf over the complex manifold $\mathfrak{h}'_{reg}$. Let ∇_1, ∇_2 be two integrable connections on E with regular singularities. If there exists an isomorphism $\hat{\psi} : (\widehat{E},\nabla_1) \to (\widehat{E},\nabla_2)$, then the local systems E^{∇_1} and E^{∇_2} are isomorphic.*

Proof. Write $\mathrm{End}(E)$ for the sheaf of endomorphisms of E. Then $\mathrm{End}(E)$ is a locally free sheaf over $\mathfrak{h}'_{reg}$. The connections ∇_1, ∇_2 define a connection ∇ on $\mathrm{End}(E)$ as follows,

$$\nabla : \mathrm{End}(E) \to \mathrm{End}(E), \quad f \mapsto \nabla_2 \circ f - f \circ \nabla_1.$$

So the isomorphism $\hat{\psi}$ is a horizontal section of $(\widehat{\mathrm{End}(E)}, \nabla)$. Let $(\mathrm{End}(E)^{\nabla})_0$ be the set of germs of horizontal sections of $(\mathrm{End}(E), \nabla)$ on zero. By the Comparison theorem [KK, Theorem 6.3.1] the canonical map $(\mathrm{End}(E)^{\nabla})_0 \to (\widehat{\mathrm{End}(E)})^{\nabla}$ is bijective. Hence there exists a holomorphic isomorphism $\psi : (E, \nabla_1) \to (E, \nabla_2)$ which maps to $\hat{\psi}$. Now, let U be an open ball in $\mathfrak{h}'_{reg}$ centered at 0 with radius ε small enough such that the holomorphic isomorphism ψ converges in U. Write E_U for the restriction of E to U. Then ψ induces an isomorphism of local systems $(E_U)^{\nabla_1} \cong (E_U)^{\nabla_2}$. Since $\mathfrak{h}'_{reg}$ is homotopy equivalent to U, we have

$$E^{\nabla_1} \cong E^{\nabla_2}.$$

□

Step 6. Finally, write I for the inclusion of $\mathrm{Proj}_c(W, \mathfrak{h})$ into $\mathcal{O}_c(W, \mathfrak{h})$. By Lemma 1.1(1) any projective object in $\mathcal{O}_c(W, \mathfrak{h})$ has a standard filtration, so (2.6) yields an isomorphism of functors

$$\mathrm{KZ}' \circ \mathrm{Res}_b \circ I \to \imath_* \circ \mathrm{KZ} \circ I.$$

Applying Lemma 1.2 to the exact functors $\mathrm{KZ}' \circ \mathrm{Res}_b$ and $\imath_* \circ \mathrm{KZ}$ yields that there is an isomorphism of functors

$$\mathrm{KZ}' \circ \mathrm{Res}_b \cong \imath_* \circ \mathrm{KZ}\,.$$

□

2.2. We give some corollaries of Theorem 2.1.

Corollary 2.3. *There is an isomorphism of functors*

$$\mathrm{KZ} \circ \mathrm{Ind}_b \cong {}^{\mathscr{H}}\mathrm{coInd}_{W'}^{W} \circ \mathrm{KZ}'\,.$$

Proof. To simplify notation let us write

$$\mathcal{O} = \mathcal{O}_c(W, \mathfrak{h}), \quad \mathcal{O}' = \mathcal{O}_{c'}(W', \overline{\mathfrak{h}}), \quad \mathscr{H} = \mathscr{H}_q(W), \quad \mathscr{H}' = \mathscr{H}_{q'}(W').$$

Recall that the functor KZ is represented by a projective object P_{KZ} in $\mathcal{O}$. So for any $N \in \mathcal{O}'$ we have a morphism of $\mathscr{H}$-modules

$$\begin{aligned} \mathrm{KZ} \circ \mathrm{Ind}_b(N) &\cong \mathrm{Hom}_{\mathcal{O}}(P_{\mathrm{KZ}}, \mathrm{Ind}_b(N)) \\ &\cong \mathrm{Hom}_{\mathcal{O}'}(\mathrm{Res}_b(P_{\mathrm{KZ}}), N) \\ &\to \mathrm{Hom}_{\mathscr{H}'}(\mathrm{KZ}'(\mathrm{Res}_b(P_{\mathrm{KZ}})), \mathrm{KZ}'(N)). \end{aligned} \tag{2.7}$$

By Theorem 2.1 we have

$$\mathrm{KZ}' \circ \mathrm{Res}_b(P_{\mathrm{KZ}}) \cong {}^{\mathscr{H}}\mathrm{Res}_{W'}^{W} \circ \mathrm{KZ}(P_{\mathrm{KZ}}).$$

Recall from Section 1.5 that the $\mathscr{H}$-module $\mathrm{KZ}(P_{\mathrm{KZ}})$ is isomorphic to $\mathscr{H}$. So as $\mathscr{H}'$-modules $\mathrm{KZ}'(\mathrm{Res}_b(P_{\mathrm{KZ}}))$ is also isomorphic to $\mathscr{H}$. Therefore the morphism (2.7) rewrites as

$$\chi(N) : \mathrm{KZ} \circ \mathrm{Ind}_b(N) \to \mathrm{Hom}_{\mathscr{H}'}(\mathscr{H}, \mathrm{KZ}'(N)). \tag{2.8}$$

It yields a morphism of functors

$$\chi : \mathrm{KZ} \circ \mathrm{Ind}_b \to {}^{\mathscr{H}}\mathrm{coInd}_{W'}^{W} \circ \mathrm{KZ}'\,.$$

Note that if N is a projective object in $\mathcal{O}'$, then $\chi(N)$ is an isomorphism by Proposition 1.4(1). So Lemma 1.2 implies that χ is an isomorphism of functors, because both functors $\mathrm{KZ} \circ \mathrm{Ind}_b$ and ${}^{\mathscr{H}}\mathrm{coInd}_{W'}^{W} \circ \mathrm{KZ}'$ are exact. □

2.3. The following lemma will be useful to us.

Lemma 2.4. *Let K, L be two right exact functors from $\mathcal{O}_1$ to $\mathcal{O}_2$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ can be either $\mathcal{O}_c(W,\mathfrak{h})$ or $\mathcal{O}_{c'}(W',\overline{\mathfrak{h}})$. Suppose that K, L map projective objects to projective ones. Then the vector space homomorphism*

$$\mathrm{Hom}(K,L)\to\mathrm{Hom}(\mathrm{KZ}_2\circ K,\mathrm{KZ}_2\circ L),\quad f\mapsto 1_{\mathrm{KZ}_2}f, \tag{2.9}$$

is an isomorphism.

Notice that if $K=L$, this is even an isomorphism of rings.

Proof. Let Proj_1, Proj_2 be respectively the subcategory of projective objects in $\mathcal{O}_1$, $\mathcal{O}_2$. Write $\tilde{K}$, $\tilde{L}$ for the functors from Proj_1 to Proj_2 given by the restrictions of K, L, respectively. Let $\mathscr{H}_2$ be the Hecke algebra corresponding to $\mathcal{O}_2$. Since the functor KZ_2 is fully faithful over Proj_2 by Proposition 1.4(1), the following functor

$$\mathrm{Fct}(\mathrm{Proj}_1,\mathrm{Proj}_2)\to\mathrm{Fct}(\mathrm{Proj}_1,\mathscr{H}_2\text{-mod}),\quad G\mapsto\mathrm{KZ}_2\circ G$$

is also fully faithful. This yields an isomorphism

$$\mathrm{Hom}(\tilde{K},\tilde{L})\xrightarrow{\sim}\mathrm{Hom}(\mathrm{KZ}_2\circ\tilde{K},\mathrm{KZ}_2\circ\tilde{L}),\quad f\mapsto 1_{\mathrm{KZ}_2}f.$$

Next, by Lemma 1.2 the canonical morphisms

$$\mathrm{Hom}(K,L)\to\mathrm{Hom}(\tilde{K},\tilde{L}),\quad \mathrm{Hom}(\mathrm{KZ}_2\circ K,\mathrm{KZ}_2\circ L)\to\mathrm{Hom}(\mathrm{KZ}_2\circ\tilde{K},\mathrm{KZ}_2\circ\tilde{L})$$

are isomorphisms. So the map (2.9) is also an isomorphism. □

Let $b(W,W'')$ be a point in $\mathfrak{h}$ whose stabilizer is W''. Let $b(W',W'')$ be its image in $\overline{\mathfrak{h}}=\mathfrak{h}/\mathfrak{h}^{W'}$ via the canonical projection. Write $b(W,W')=b$.

Corollary 2.5. *There are isomorphisms of functors*

$$\begin{aligned}\mathrm{Res}_{b(W',W'')}\circ\mathrm{Res}_{b(W,W')} &\cong \mathrm{Res}_{b(W,W'')},\\ \mathrm{Ind}_{b(W,W')}\circ\mathrm{Ind}_{b(W',W'')} &\cong \mathrm{Ind}_{b(W,W'')}\,.\end{aligned}$$

Proof. Since the restriction functors map projective objects to projective ones by Proposition 1.6(1), Lemma 2.4 applied to the categories $\mathcal{O}_1=\mathcal{O}_c(W,\mathfrak{h})$, $\mathcal{O}_2=\mathcal{O}_{c''}(W'',\mathfrak{h}/\mathfrak{h}^{W''})$ yields an isomorphism

$$\begin{aligned}&\mathrm{Hom}(\mathrm{Res}_{b(W',W'')}\circ\mathrm{Res}_{b(W,W')},\mathrm{Res}_{b(W,W'')})\\ &\cong\mathrm{Hom}(\mathrm{KZ}''\circ\mathrm{Res}_{b(W',W'')}\circ\mathrm{Res}_{b(W,W')},\mathrm{KZ}''\circ\mathrm{Res}_{b(W,W'')}).\end{aligned}$$

By Theorem 2.1 the set on the second row is

$$\mathrm{Hom}({}^{\mathscr{H}}\mathrm{Res}^{W'}_{W''}\circ{}^{\mathscr{H}}\mathrm{Res}^{W}_{W'}\circ\mathrm{KZ},\ {}^{\mathscr{H}}\mathrm{Res}^{W}_{W''}\circ\mathrm{KZ}). \tag{2.10}$$

By the presentations of Hecke algebras in [BMR, Proposition 4.22], there is an isomorphism

$$\sigma:\ {}^{\mathscr{H}}\mathrm{Res}^{W'}_{W''}\circ{}^{\mathscr{H}}\mathrm{Res}^{W}_{W'}\xrightarrow{\sim}{}^{\mathscr{H}}\mathrm{Res}^{W}_{W''}\,.$$

Hence the element $\sigma 1_{\mathrm{KZ}}$ in the set (2.10) maps to an isomorphism

$$\mathrm{Res}_{b(W',W'')}\circ\mathrm{Res}_{b(W,W')}\cong\mathrm{Res}_{b(W,W'')}\,.$$

This proves the first isomorphism in the corollary. The second one follows from the uniqueness of right adjoint functor. □

2.4. **Biadjointness of Res_b and Ind_b.** Recall that a finite dimensional $\mathbb{C}$-algebra A is symmetric if A is isomorphic to $A^* = \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C})$ as (A, A)-bimodules.

Lemma 2.6. *Assume that $\mathscr{H}_q(W)$ and $\mathscr{H}_{q'}(W')$ are symmetric algebras. Then the functors ${}^{\mathscr{H}}\mathrm{Ind}_{W'}^{W}$ and ${}^{\mathscr{H}}\mathrm{coInd}_{W'}^{W}$ are isomorphic, i.e., the functor ${}^{\mathscr{H}}\mathrm{Ind}_{W'}^{W}$ is biadjoint to ${}^{\mathscr{H}}\mathrm{Res}_{W'}^{W}$.*

Proof. We abbreviate $\mathscr{H} = \mathscr{H}_q(W)$ and $\mathscr{H}' = \mathscr{H}_{q'}(W')$. Since $\mathscr{H}$ is free as a left $\mathscr{H}'$-module, for any $\mathscr{H}'$-module M the map

$$\mathrm{Hom}_{\mathscr{H}'}(\mathscr{H}, \mathscr{H}') \otimes_{\mathscr{H}'} M \to \mathrm{Hom}_{\mathscr{H}'}(\mathscr{H}, M) \tag{2.11}$$

given by multiplication is an isomorphism of $\mathscr{H}$-modules. By assumption $\mathscr{H}'$ is isomorphic to $(\mathscr{H}')^*$ as $(\mathscr{H}', \mathscr{H}')$-bimodules. Thus we have the following $(\mathscr{H}, \mathscr{H}')$-bimodule isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathscr{H}'}(\mathscr{H}, \mathscr{H}') &\cong \mathrm{Hom}_{\mathscr{H}'}(\mathscr{H}, (\mathscr{H}')^*) \\ &\cong \mathrm{Hom}_{\mathbb{C}}(\mathscr{H}' \otimes_{\mathscr{H}'} \mathscr{H}, \mathbb{C}) \\ &\cong \mathscr{H}^* \\ &\cong \mathscr{H}. \end{aligned}$$

The last isomorphism follows from the fact the $\mathscr{H}$ is symmetric. Thus, by (2.11) the functors ${}^{\mathscr{H}}\mathrm{Ind}_{W'}^{W}$ and ${}^{\mathscr{H}}\mathrm{coInd}_{W'}^{W}$ are isomorphic. □

Remark 2.7. It is proved that $\mathscr{H}_q(W)$ is a symmetric algebra for all irreducible complex reflection group W except for some of the 34 exceptional groups in the Shephard-Todd classification. See [BMM, Section 2A] for details.

The biadjointness of Res_b and Ind_b was conjectured in [BE, Remark 3.18] and was announced by I. Gordon and M. Martino. We give a proof in Proposition 2.9 since it seems not yet to be available in the literature. Let us first consider the following lemma.

Lemma 2.8. *(1) Let A, B be noetherian algebras and T be a functor*

$$T : A\text{-mod} \to B\text{-mod}.$$

If T is right exact and commutes with direct sums, then it has a right adjoint.
(2) The functor

$$\mathrm{Res}_b : \mathcal{O}_c(W, \mathfrak{h}) \to \mathcal{O}_{c'}(W', \overline{\mathfrak{h}})$$

has a left adjoint.

Proof. (1) Consider the (B, A)-bimodule $M = T(A)$. We claim that the functor T is isomorphic to the functor $M \otimes_A -$. Indeed, by definition we have $T(A) \cong M \otimes_A A$ as B modules. Now, for any $N \in A$-mod, since N is finitely generated and A is noetherian there exists m, $n \in \mathbb{N}$ and an exact sequence

$$A^{\oplus n} \longrightarrow A^{\oplus m} \longrightarrow N \longrightarrow 0.$$

Since both T and $M \otimes_A -$ are right exact and they commute with direct sums, the fact that $T(A) \cong M \otimes_A A$ implies that $T(N) \cong M \otimes_A N$ as B-modules. This proved the claim. Now, the functor $M \otimes_A -$ has a right adjoint $\mathrm{Hom}_B(M, -)$, so T also has a right adjoint.

(2) Recall that for any complex reflection group W, a contravariant duality functor

$$(-)^\vee : \mathcal{O}_c(W, \mathfrak{h}) \to \mathcal{O}_{c^\dagger}(W, \mathfrak{h}^*)$$

was defined in [GGOR, Section 4.2], here $c^\dagger : \mathcal{S} \to \mathbb{C}$ is another parameter explicitly determined by c. Consider the functor

$$\mathrm{Res}_b^\vee = (-)^\vee \circ \mathrm{Res}_b \circ (-)^\vee : \mathcal{O}_{c^\dagger}(W, \mathfrak{h}^*) \to \mathcal{O}_{c'^\dagger}(W', (\overline{\mathfrak{h}})^*).$$

The category $\mathcal{O}_{c^\dagger}(W,\mathfrak{h}^*)$ has a projective generator P. The algebra $\mathrm{End}_{\mathcal{O}_{c^\dagger}(W,\mathfrak{h}^*)}(P)^{\mathrm{op}}$ is finite dimensional over $\mathbb{C}$ and by Morita theory we have an equivalence of categories

$$\mathcal{O}_{c^\dagger}(W,\mathfrak{h}^*)\cong \mathrm{End}_{\mathcal{O}_{c^\dagger}(W,\mathfrak{h}^*)}(P)^{\mathrm{op}}\text{-mod}\,.$$

Since the functor $\mathrm{Res}_b^\vee$ is exact and obviously commutes with direct sums, by part (1) it has a right adjoint Ψ. Then it follows that $(-)^\vee\circ\Psi\circ(-)^\vee$ is left adjoint to Res_b. The lemma is proved. □

Proposition 2.9. *Under the assumption of Lemma 2.6, the functor* Ind_b *is left adjoint to* Res_b*.*

Proof. Step 1. We abbreviate $\mathcal{O}=\mathcal{O}_c(W,\mathfrak{h})$, $\mathcal{O}'=\mathcal{O}_{c'}(W',\overline{\mathfrak{h}})$, $\mathscr{H}=\mathscr{H}_q(W)$, $\mathscr{H}'=\mathscr{H}_{q'}(W')$, and write $\mathrm{Id}_{\mathcal{O}}$, $\mathrm{Id}_{\mathcal{O}'}$, $\mathrm{Id}_{\mathscr{H}}$, $\mathrm{Id}_{\mathscr{H}'}$ for the identity functor on the corresponding categories. We also abbreviate $E^{\mathscr{H}}={}^{\mathscr{H}}\mathrm{Res}^W_{W'}$, $F^{\mathscr{H}}={}^{\mathscr{H}}\mathrm{Ind}^W_{W'}$ and $E=\mathrm{Res}_b$. By Lemma 2.8 the functor E has a left adjoint. We denote it by $F:\mathcal{O}'\to\mathcal{O}$. Recall the functors

$$\mathrm{KZ}:\mathcal{O}\to\mathscr{H}\text{-mod},\quad \mathrm{KZ}':\mathcal{O}'\to\mathscr{H}'\text{-mod}\,.$$

The goal of this step is to show that there exists an isomorphism of functors

$$\mathrm{KZ}\circ F\cong F^{\mathscr{H}}\circ\mathrm{KZ}'\,.$$

To this end, let S, S' be respectively the right adjoints of KZ, KZ$'$, see Section 1.5. We will first give an isomorphism of functors

$$F^{\mathscr{H}}\cong \mathrm{KZ}\circ F\circ S'.$$

Let $M\in\mathscr{H}'$-mod and $N\in\mathscr{H}$-mod. Consider the following equalities given by adjunctions

$$\begin{aligned}\mathrm{Hom}_{\mathscr{H}}(\mathrm{KZ}\circ F\circ S'(M),N) &= \mathrm{Hom}_{\mathcal{O}}(F\circ S'(M),S(N))\\ &= \mathrm{Hom}_{\mathcal{O}'}(S'(M),E\circ S(N)).\end{aligned}$$

The functor KZ$'$ yields a map

$$a(M,N):\mathrm{Hom}_{\mathcal{O}'}(S'(M),E\circ S(N))\to\mathrm{Hom}_{\mathscr{H}'}(\mathrm{KZ}'\circ S'(M),\mathrm{KZ}'\circ E\circ S(N)). \tag{2.12}$$

Since the canonical adjunction maps $\mathrm{KZ}'\circ S'\to\mathrm{Id}_{\mathscr{H}'}$, $\mathrm{KZ}\circ S\to\mathrm{Id}_{\mathscr{H}}$ are isomorphisms (see Section 1.5) and since we have an isomorphism of functors $\mathrm{KZ}'\circ E\cong E^{\mathscr{H}}\circ\mathrm{KZ}$ by Theorem 2.1, we get the following equalities

$$\begin{aligned}\mathrm{Hom}_{\mathscr{H}'}(\mathrm{KZ}'\circ S'(M),\mathrm{KZ}'\circ E\circ S(N)) &= \mathrm{Hom}_{\mathscr{H}'}(M,E^{\mathscr{H}}\circ\mathrm{KZ}\circ S(N))\\ &= \mathrm{Hom}_{\mathscr{H}'}(M,E^{\mathscr{H}}(N))\\ &= \mathrm{Hom}_{\mathscr{H}}(F^{\mathscr{H}}(M),N).\end{aligned}$$

In the last equality we used that $F^{\mathscr{H}}$ is left adjoint to $E^{\mathscr{H}}$. So the map (2.12) can be rewritten into the following form

$$a(M,N):\mathrm{Hom}_{\mathscr{H}}(\mathrm{KZ}\circ F\circ S'(M),N)\to\mathrm{Hom}_{\mathscr{H}}(F^{\mathscr{H}}(M),N).$$

Now, take $N=\mathscr{H}$. Recall that $\mathscr{H}$ is isomorphic to $\mathrm{KZ}(P_{\mathrm{KZ}})$ as $\mathscr{H}$-modules. Since P_{KZ} is projective, by Proposition 1.4(2) we have a canonical isomorphism in $\mathcal{O}$

$$P_{\mathrm{KZ}}\cong S(\mathrm{KZ}(P_{\mathrm{KZ}}))=S(\mathscr{H}).$$

Further E maps projectives to projectives by Proposition 1.6(1), so $E\circ S(\mathscr{H})$ is also projective. Hence Proposition 1.4(1) implies that in this case (2.12) is an isomorphism for any M, i.e., we get an isomorphism

$$a(M,\mathscr{H}):\mathrm{Hom}_{\mathscr{H}}(\mathrm{KZ}\circ F\circ S'(M),\mathscr{H})\xrightarrow{\sim}\mathrm{Hom}_{\mathscr{H}}(F^{\mathscr{H}}(M),\mathscr{H}).$$

Further this is an isomorphism of right $\mathscr{H}$-modules with respect to the $\mathscr{H}$-actions induced by the right action of $\mathscr{H}$ on itself. Now, the fact that $\mathscr{H}$ is a symmetric algebra yields that for any finite dimensional $\mathscr{H}$-module N we have isomorphisms of right $\mathscr{H}$-modules

$$\begin{aligned}\mathrm{Hom}_{\mathscr{H}}(N,\mathscr{H}) &\cong \mathrm{Hom}_{\mathscr{H}}(N,\mathrm{Hom}_{\mathbb{C}}(\mathscr{H},\mathbb{C}))\\ &\cong \mathrm{Hom}_{\mathbb{C}}(N,\mathbb{C}).\end{aligned}$$

Therefore $a(M,\mathscr{H})$ yields an isomorphism of right $\mathscr{H}$-modules

$$\mathrm{Hom}_{\mathbb{C}}(\mathrm{KZ}\circ F\circ S'(M),\mathbb{C})\to \mathrm{Hom}_{\mathbb{C}}(F^{\mathscr{H}}(M),\mathbb{C}).$$

We deduce a natural isomorphism of left $\mathscr{H}$-modules

$$\mathrm{KZ}\circ F\circ S'(M)\cong F^{\mathscr{H}}(M)$$

for any $\mathscr{H}'$-module M. This gives an isomorphism of functors

$$\psi:\mathrm{KZ}\circ F\circ S'\xrightarrow{\sim} F^{\mathscr{H}}.$$

Finally, consider the canonical adjunction map $\eta:\mathrm{Id}_{\mathcal{O}'}\to S'\circ\mathrm{KZ}'$. We have a morphism of functors

$$\phi=(1_{\mathrm{KZ}\circ F}\eta)\circ(\psi 1_{\mathrm{KZ}'}):\mathrm{KZ}\circ F\to F^{\mathscr{H}}\circ\mathrm{KZ}'.$$

Note that $\psi 1_{\mathrm{KZ}'}$ is an isomorphism of functors. If Q is a projective object in $\mathcal{O}'$, then by Proposition 1.4(2) the morphism $\eta(Q):Q\to S'\circ\mathrm{KZ}'(Q)$ is also an isomorphism, so $\phi(Q)$ is an isomorphism. This implies that ϕ is an isomorphism of functors by Lemma 1.2, because both $\mathrm{KZ}\circ F$ and $F^{\mathscr{H}}\circ\mathrm{KZ}'$ are right exact functors. Here the right exactness of F follows from that it is left adjoint to E. So we get the desired isomorphism of functors

$$\mathrm{KZ}\circ F\cong F^{\mathscr{H}}\circ\mathrm{KZ}'.$$

Step 2. Let us now prove that F is right adjoint to E. By uniqueness of adjoint functors, this will imply that F is isomorphic to Ind_b. First, by Lemma 2.6 the functor $F^{\mathscr{H}}$ is isomorphic to ${}^{\mathscr{H}}\mathrm{coInd}_{W'}^{W}$. So $F^{\mathscr{H}}$ is right adjoint to $E^{\mathscr{H}}$, i.e., we have morphisms of functors

$$\varepsilon^{\mathscr{H}}:E^{\mathscr{H}}\circ F^{\mathscr{H}}\to\mathrm{Id}_{\mathscr{H}'},\quad \eta^{\mathscr{H}}:\mathrm{Id}_{\mathscr{H}}\to F^{\mathscr{H}}\circ E^{\mathscr{H}}$$

such that

$$(\varepsilon^{\mathscr{H}}1_{E^{\mathscr{H}}})\circ(1_{E^{\mathscr{H}}}\eta^{\mathscr{H}})=1_{E^{\mathscr{H}}},\quad (1_{F^{\mathscr{H}}}\varepsilon^{\mathscr{H}})\circ(\eta^{\mathscr{H}}1_{F^{\mathscr{H}}})=1_{F^{\mathscr{H}}}.$$

Next, both F and E have exact right adjoints, given respectively by E and Ind_b. Therefore F and E map projective objects to projective ones. Applying Lemma 2.4 to $\mathcal{O}_1=\mathcal{O}_2=\mathcal{O}'$, $K=E\circ F$, $L=\mathrm{Id}_{\mathcal{O}'}$ yields that the following map is bijective

$$\mathrm{Hom}(E\circ F,\mathrm{Id}_{\mathcal{O}'})\to\mathrm{Hom}(\mathrm{KZ}'\circ E\circ F,\mathrm{KZ}'\circ\mathrm{Id}_{\mathcal{O}}),\quad f\mapsto 1_{\mathrm{KZ}'}f. \tag{2.13}$$

By Theorem 2.1 and Step 1 there exist isomorphisms of functors

$$\phi_E:E^{\mathscr{H}}\circ\mathrm{KZ}\xrightarrow{\sim}\mathrm{KZ}'\circ E,\quad \phi_F:F^{\mathscr{H}}\circ\mathrm{KZ}'\xrightarrow{\sim}\mathrm{KZ}\circ F.$$

Let

$$\begin{aligned}\phi_{EF}&=(\phi_E 1_F)\circ(1_{E^{\mathscr{H}}}\phi_F):E^{\mathscr{H}}\circ F^{\mathscr{H}}\circ\mathrm{KZ}'\xrightarrow{\sim}\mathrm{KZ}'\circ E\circ F,\\ \phi_{FE}&=(\phi_F 1_E)\circ(1_{F^{\mathscr{H}}}\phi_E):F^{\mathscr{H}}\circ E^{\mathscr{H}}\circ\mathrm{KZ}\xrightarrow{\sim}\mathrm{KZ}\circ F\circ E.\end{aligned}$$

Identify

$$\mathrm{KZ}\circ\mathrm{Id}_{\mathcal{O}}=\mathrm{Id}_{\mathscr{H}}\circ\mathrm{KZ},\quad \mathrm{KZ}'\circ\mathrm{Id}_{\mathcal{O}'}=\mathrm{Id}_{\mathscr{H}'}\circ\mathrm{KZ}'.$$

We have a bijective map

$$\mathrm{Hom}(\mathrm{KZ}'\circ E\circ F,\mathrm{KZ}'\circ\mathrm{Id}_{\mathcal{O}'})\xrightarrow{\sim}\mathrm{Hom}(E^{\mathscr{H}}\circ F^{\mathscr{H}}\circ\mathrm{KZ}',\mathrm{Id}_{\mathscr{H}'}\circ\mathrm{KZ}'),\quad g\mapsto g\circ\phi_{EF}.$$

Together with (2.13), it implies that there exists a unique morphism $\varepsilon : E\circ F \to \mathrm{Id}_{\mathcal{O}'}$ such that

$$(1_{\mathrm{KZ}'}\varepsilon)\circ\phi_{EF} = \varepsilon^{\mathscr{H}} 1_{\mathrm{KZ}'}.$$

Similarly, there exists a unique morphism $\eta : \mathrm{Id}_{\mathcal{O}} \to F\circ E$ such that

$$(\phi_{FE})^{-1}\circ(1_{\mathrm{KZ}}\eta) = \eta^{\mathscr{H}} 1_{\mathrm{KZ}}.$$

Now, we have the following commutative diagram

$$\begin{tikzcd}
E^{\mathscr{H}}\circ \mathrm{KZ} \arrow[r, equal] \arrow[d, "1_{E^{\mathscr{H}}}\eta^{\mathscr{H}}1_{\mathrm{KZ}}"'] & E^{\mathscr{H}}\circ \mathrm{KZ} \arrow[r, "\phi_E"] \arrow[d, "1_{E^{\mathscr{H}}}1_{\mathrm{KZ}}\eta"] & \mathrm{KZ}'\circ E \arrow[d, "1_{\mathrm{KZ}'}1_E\eta"] \\
E^{\mathscr{H}}\circ F^{\mathscr{H}}\circ E^{\mathscr{H}}\circ \mathrm{KZ} \arrow[r, "1_{E^{\mathscr{H}}}\phi_{FE}"] \arrow[d, equal] & E^{\mathscr{H}}\circ \mathrm{KZ}\circ F\circ E \arrow[r, "\phi_E 1_F 1_E"] & \mathrm{KZ}'\circ E\circ F\circ E \arrow[d, equal] \\
E^{\mathscr{H}}\circ F^{\mathscr{H}}\circ E^{\mathscr{H}}\circ \mathrm{KZ} \arrow[r, "1_{E^{\mathscr{H}}}1_{F^{\mathscr{H}}}\phi_E"] \arrow[d, "\varepsilon^{\mathscr{H}}1_{E^{\mathscr{H}}}1_{\mathrm{KZ}}"'] & E^{\mathscr{H}}\circ F^{\mathscr{H}}\circ \mathrm{KZ}'\circ E \arrow[r, "\phi_{EF}1_E"] \arrow[u, "1_{E^{\mathscr{H}}}\phi_F 1_E"'] \arrow[d, "\varepsilon^{\mathscr{H}}1_{\mathrm{KZ}'}1_E"] & \mathrm{KZ}'\circ E\circ F\circ E \arrow[d, "1_{\mathrm{KZ}'}\varepsilon 1_E"] \\
E^{\mathscr{H}}\circ \mathrm{KZ} \arrow[r, "\phi_E"] & \mathrm{KZ}'\circ E \arrow[r, equal] & \mathrm{KZ}'\circ E.
\end{tikzcd}$$

It yields that

$$(1_{\mathrm{KZ}'}\varepsilon 1_E)\circ(1_{\mathrm{KZ}'}1_E\eta) = \phi_E\circ(\varepsilon^{\mathscr{H}}1_{E^{\mathscr{H}}}1_{\mathrm{KZ}})\circ(1_{E^{\mathscr{H}}}\eta^{\mathscr{H}}1_{\mathrm{KZ}})\circ(\phi_E)^{-1}.$$

We deduce that

$$\begin{aligned} 1_{\mathrm{KZ}'}((\varepsilon 1_E)\circ(1_E\eta)) &= \phi_E\circ(1_{E^{\mathscr{H}}}1_{\mathrm{KZ}})\circ(\phi_E)^{-1} \\ &= 1_{\mathrm{KZ}'}1_E. \end{aligned} \tag{2.14}$$

By applying Lemma 2.4 to $\mathcal{O}_1 = \mathcal{O}$, $\mathcal{O}_2 = \mathcal{O}'$, $K = L = E$, we deduce that the following map is bijective

$$\mathrm{End}(E)\to\mathrm{End}(\mathrm{KZ}'\circ E),\quad f\mapsto 1_{\mathrm{KZ}'}f.$$

Hence (2.14) implies that

$$(\varepsilon 1_E)\circ(1_E\eta) = 1_E.$$

Similarly, we have $(1_F\varepsilon)\circ(\eta 1_F) = 1_F$. So E is left adjoint to F. By uniqueness of adjoint functors this implies that F is isomorphic to Ind_b. Therefore Ind_b is biadjoint to Res_b. $\square$

3. Reminders on the Cyclotomic case.

From now on we will concentrate on the cyclotomic rational DAHA's. We fix some notation in this section.

3.1. Let l, n be positive integers. Write $\varepsilon = \exp(\frac{2\pi\sqrt{-1}}{l})$. Let $\mathfrak{h} = \mathbb{C}^n$, write $\{y_1,\ldots,y_n\}$ for its standard basis. For $1\leqslant i,j,k\leqslant n$ with i,j,k distinct, let ε_k, s_{ij} be the following elements of $GL(\mathfrak{h})$:

$$\varepsilon_k(y_k) = \varepsilon y_k,\quad \varepsilon_k(y_j) = y_j,\quad s_{ij}(y_i) = y_j,\quad s_{ij}(y_k) = y_k.$$

Let $B_n(l)$ be the subgroup of $GL(\mathfrak{h})$ generated by ε_k and s_{ij} for $1\leqslant k\leqslant n$ and $1\leqslant i<j\leqslant n$. It is a complex reflection group with the set of reflections

$$\mathcal{S}_n = \{\varepsilon_i^p : 1\leqslant i\leqslant n, 1\leqslant p\leqslant l-1\}\bigsqcup\{s_{ij}^{(p)} = s_{ij}\varepsilon_i^p\varepsilon_j^{-p} : 1\leqslant i<j\leqslant n, 1\leqslant p\leqslant l\}.$$

Note that there is an obvious inclusion $\mathcal{S}_{n-1}\hookrightarrow\mathcal{S}_n$. It yields an embedding

$$B_{n-1}(l)\hookrightarrow B_n(l). \tag{3.1}$$

This embedding identifies $B_{n-1}(l)$ with the parabolic subgroup of $B_n(l)$ given by the stabilizer of the point $b_n = (0,\ldots,0,1)\in\mathbb{C}^n$.

The cyclotomic rational DAHA is the algebra $H_c(B_n(l),\mathfrak{h})$. We will use another presentation in which we replace the parameter c by an l-tuple $\mathbf{h}=(h,h_1,\dots,h_{l-1})$ such that

$$c_{s_{ij}^{(p)}}=-h,\quad c_{\varepsilon_p}=\frac{-1}{2}\sum_{p'=1}^{l-1}(\varepsilon^{-pp'}-1)h_{p'}.$$

We will denote $H_c(B_n(l),\mathfrak{h})$ by $H_{\mathbf{h},n}$. The corresponding category $\mathcal{O}$ will be denoted by $\mathcal{O}_{\mathbf{h},n}$. In the rest of the paper, we will fix the positive integer l. We will also fix a positive integer $e\geqslant 2$ and an l-tuple of integers $\mathbf{s}=(s_1,\dots,s_l)$. *We will always assume that the parameter* $\mathbf{h}$ *is given by the following formulas ,*

$$h=\frac{-1}{e},\quad h_p=\frac{s_{p+1}-s_p}{e}-\frac{1}{l},\quad 1\leqslant p\leqslant l-1\,. \tag{3.2}$$

The functor $\mathrm{KZ}(B_n(l),\mathbb{C}^n)$ goes from $\mathcal{O}_{\mathbf{h},n}$ to the category of finite dimensional modules of a certain Hecke algebra $\mathscr{H}_{\mathbf{q},n}$ attached to the group $B_n(l)$. Here the parameter is $\mathbf{q}=(q,q_1,\dots,q_l)$ with

$$q=\exp(2\pi\sqrt{-1}/e),\quad q_p=q^{s_p},\quad 1\leqslant p\leqslant l.$$

The algebra $\mathscr{H}_{\mathbf{q},n}$ has the following presentation:

- Generators: $T_0,T_1,\dots,T_{n-1}$,
- Relations:

$$\begin{gathered}(T_0-q_1)\cdots(T_0-q_l)=(T_i+1)(T_i-q)=0,\quad 1\leqslant i\leqslant n-1,\\ T_0T_1T_0T_1=T_1T_0T_1T_0,\\ T_iT_j=T_jT_i,\quad \text{if } |i-j|>1,\\ T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1},\quad 1\leqslant i\leqslant n-2.\end{gathered}$$

The algebra $\mathscr{H}_{\mathbf{q},n}$ satisfies the assumption of Section 2, i.e., it has the same dimension as $\mathbb{C}B_n(l)$.

3.2. For each positive integer n, the embedding (3.1) of $B_n(l)$ into $B_{n+1}(l)$ yields an embedding of Hecke algebras

$$\imath_{\mathbf{q}}:\mathscr{H}_{\mathbf{q},n}\hookrightarrow\mathscr{H}_{\mathbf{q},n+1},$$

see Section 1.2. Under the presentation above this embedding is given by

$$\imath_{\mathbf{q}}(T_i)=T_i,\quad \forall\ 0\leqslant i\leqslant n-1,$$

see [BMR, Proposition 2.29].

We will consider the following restriction and induction functors:

$$\begin{aligned}E(n)=\mathrm{Res}_{b_n},\quad E(n)^{\mathscr{H}}&={}^{\mathscr{H}}\mathrm{Res}_{B_{n-1}(l)}^{B_n(l)},\\ F(n)=\mathrm{Ind}_{b_n},\quad F(n)^{\mathscr{H}}&={}^{\mathscr{H}}\mathrm{Ind}_{B_{n-1}(l)}^{B_n(l)}\,.\end{aligned}$$

The algebra $\mathscr{H}_{\mathbf{q},n}$ is symmetric (see Remark 2.7). Hence by Lemma 2.6 we have

$$F(n)^{\mathscr{H}}\cong{}^{\mathscr{H}}\mathrm{coInd}_{B_{n-1}(l)}^{B_n(l)}\,.$$

We will abbreviate

$$\mathcal{O}_{\mathbf{h},\mathbb{N}}=\bigoplus_{n\in\mathbb{N}}\mathcal{O}_{\mathbf{h},n},\quad \mathrm{KZ}=\bigoplus_{n\in\mathbb{N}}\mathrm{KZ}(B_n(l),\mathbb{C}^n),\quad \mathscr{H}_{\mathbf{q},\mathbb{N}}\text{-mod}=\bigoplus_{n\in\mathbb{N}}\mathscr{H}_{\mathbf{q},n}\text{-mod}\,.$$

So KZ is the Knizhnik-Zamolodchikov functor from $\mathcal{O}_{\mathbf{h},\mathbb{N}}$ to $\mathscr{H}_{\mathbf{q},\mathbb{N}}$-mod. Let

$$E = \bigoplus_{n \geqslant 1} E(n), \quad E^{\mathscr{H}} = \bigoplus_{n \geqslant 1} E^{\mathscr{H}}(n),$$
$$F = \bigoplus_{n \geqslant 1} F(n), \quad F^{\mathscr{H}} = \bigoplus_{n \geqslant 1} F^{\mathscr{H}}(n).$$

So $(E^{\mathscr{H}}, F^{\mathscr{H}})$ is a pair of biadjoint endo-functors of $\mathscr{H}_{\mathbf{q},\mathbb{N}}$-mod, and (E, F) is a pair of biadjoint endo-functors of $\mathcal{O}_{\mathbf{h},\mathbb{N}}$ by Proposition 2.9.

3.3. **Fock spaces.** Recall that an l-partition is an l-tuple $\lambda = (\lambda^1, \cdots, \lambda^l)$ with each λ^j a partition, that is a sequence of integers $(\lambda^j)_1 \geqslant \cdots \geqslant (\lambda^j)_k > 0$. To any l-partition $\lambda = (\lambda^1, \ldots, \lambda^l)$ we attach the set

$$\Upsilon_\lambda = \{(a, b, j) \in \mathbb{N} \times \mathbb{N} \times (\mathbb{Z}/l\mathbb{Z}) : 0 < b \leqslant (\lambda^j)_a\}.$$

Write $|\lambda|$ for the number of elements in this set, we say that λ is an l-partition of $|\lambda|$. For $n \in \mathbb{N}$ we denote by $\mathcal{P}_{n,l}$ the set of l-partitions of n. For any l-partition μ such that Υ_μ contains Υ_λ, we write μ/λ for the complement of Υ_λ in Υ_μ. Let $|\mu/\lambda|$ be the number of elements in this set. To each element (a, b, j) in Υ_λ we attach an element

$$\mathrm{res}((a, b, j)) = b - a + s_j \in \mathbb{Z}/e\mathbb{Z},$$

called the residue of (a, b, j). Here s_j is the j-th component of our fixed l-tuple $\mathbf{s}$.

The Fock space with multi-charge $\mathbf{s}$ is the $\mathbb{C}$-vector space $\mathcal{F}_{\mathbf{s}}$ spanned by the l-partitions, i.e.,

$$\mathcal{F}_{\mathbf{s}} = \bigoplus_{n \in \mathbb{N}} \bigoplus_{\lambda \in \mathcal{P}_{n,l}} \mathbb{C}\lambda.$$

It admits an integrable $\widehat{\mathfrak{sl}}_e$-module structure with the Chevalley generators acting as follows (cf. [JMMO]): for any $i \in \mathbb{Z}/e\mathbb{Z}$,

$$e_i(\lambda) = \sum_{|\lambda/\mu|=1, \mathrm{res}(\lambda/\mu)=i} \mu, \quad f_i(\lambda) = \sum_{|\mu/\lambda|=1, \mathrm{res}(\mu/\lambda)=i} \mu. \tag{3.3}$$

For each $n \in \mathbb{Z}$ set $\Lambda_n = \Lambda_{\underline{n}}$, where $\underline{n}$ is the image of n in $\mathbb{Z}/e\mathbb{Z}$ and $\Lambda_{\underline{n}}$ is the corresponding fundamental weight of $\widehat{\mathfrak{sl}}_e$. Set

$$\Lambda_{\mathbf{s}} = \Lambda_{\underline{s_1}} + \cdots + \Lambda_{\underline{s_l}}.$$

Each l-partition λ is a weight vector of $\mathcal{F}_{\mathbf{s}}$ with weight

$$\mathrm{wt}(\lambda) = \Lambda_{\mathbf{s}} - \sum_{i \in \mathbb{Z}/e\mathbb{Z}} n_i \alpha_i, \tag{3.4}$$

where n_i is the number of elements in the set $\{(a, b, j) \in \Upsilon_\lambda : \mathrm{res}((a, b, j)) = i\}$. We will call $\mathrm{wt}(\lambda)$ the weight of λ.

In [R1, Section 6.1.1] an explicit bijection was given between the sets $\mathrm{Irr}(B_n(l))$ and $\mathcal{P}_{n,l}$. Using this bijection we identify these two sets and index the standard and simple modules in $\mathcal{O}_{\mathbf{h},\mathbb{N}}$ by l-partitions. In particular, we have an isomorphism of $\mathbb{C}$-vector spaces

$$\theta : K(\mathcal{O}_{\mathbf{h},\mathbb{N}}) \xrightarrow{\sim} \mathcal{F}_{\mathbf{s}}, \quad [\Delta(\lambda)] \mapsto \lambda. \tag{3.5}$$

3.4. We end this section by the following lemma. Recall that the functor KZ gives a map $K(\mathcal{O}_{\mathbf{h},n}) \to K(\mathscr{H}_{\mathbf{q},n})$. For any l-partition λ of n let S_λ be the corresponding Specht module in $\mathscr{H}_{\mathbf{q},n}$-mod, see [A2, Definition 13.22] for its definition.

Lemma 3.1. *In $K(\mathscr{H}_{\mathbf{q},n})$, we have* $\mathrm{KZ}([\Delta(\lambda)]) = [S_\lambda]$.

Proof. Let R be any commutative ring over $\mathbb{C}$. For any l-tuplet $\mathbf{z} = (z, z_1, \ldots, z_{l-1})$ of elements in R one defines the rational DAHA over R attached to $B_n(l)$ with parameter $\mathbf{z}$ in the same way as before. Denote it by $H_{R,\mathbf{z},n}$. The standard modules $\Delta_R(\lambda)$ are also defined as before. For any $(l+1)$-tuplet $\mathbf{u} = (u, u_1, \ldots, u_l)$ of invertible elements in R the Hecke algebra $\mathscr{H}_{R,\mathbf{u},n}$ over R attached to $B_n(l)$ with parameter $\mathbf{u}$ is defined by the same presentation as in Section 3.1. The Specht modules $S_{R,\lambda}$ are also well-defined (see [A2]). If R is a field, we will write $\mathrm{Irr}(\mathscr{H}_{R,\mathbf{u},n})$ for the set of isomorphism classes of simple $\mathscr{H}_{R,\mathbf{u},n}$-modules.

Now, fix R to be the ring of holomorphic functions of one variable ϖ. We choose $\mathbf{z} = (z, z_1, \ldots, z_{l-1})$ to be given by

$$z = l\varpi, \quad z_p = (s_{p+1} - s_p)l\varpi + e\varpi, \quad 1 \leqslant p \leqslant l-1.$$

Write $x = \exp(-2\pi\sqrt{-1}\varpi)$. Let $\mathbf{u} = (u, u_1, \ldots, u_l)$ be given by

$$u = x^l, \quad u_p = \varepsilon^{p-1} x^{s_p l - (p-1)e}, \quad 1 \leqslant p \leqslant l.$$

By [BMR, Theorem 4.12] the same definition as in Section 1.5 yields a well defined $\mathscr{H}_{R,\mathbf{u},n}$-module

$$T_R(\lambda) = \mathrm{KZ}_R(\Delta_R(\lambda)).$$

It is a free R-module of finite rank and it commutes with the base change functor by the existence and unicity theorem for linear differential equations, i.e., for any ring homomorphism $R \to R'$ over $\mathbb{C}$, we have a canonical isomorphism of $\mathscr{H}_{R',\mathbf{u},n}$-modules

$$T_{R'}(\lambda) = \mathrm{KZ}_{R'}(\Delta_{R'}(\lambda)) \cong T_R(\lambda) \otimes_R R'. \tag{3.6}$$

In particular, for any ring homomorphism $a : R \to \mathbb{C}$. Write $\mathbb{C}_a$ for the vector space $\mathbb{C}$ equipped with the R-module structure given by a. Let $a(\mathbf{z})$, $a(\mathbf{u})$ denote the images of $\mathbf{z}$, $\mathbf{u}$ by a. Note that we have $H_{a(\mathbf{z}),n} = H_{R,\mathbf{z},n} \otimes_R \mathbb{C}_a$ and $\mathscr{H}_{a(\mathbf{u}),n} = \mathscr{H}_{R,\mathbf{u},n} \otimes_R \mathbb{C}_a$. Denote the Knizhnik-Zamolodchikov functor of $H_{a(\mathbf{z}),n}$ by $\mathrm{KZ}_{a(\mathbf{z})}$ and the standard module corresponding to λ by $\Delta_{a(\mathbf{z})}(\lambda)$. Then we have an isomorphism of $\mathscr{H}_{a(\mathbf{u}),n}$-modules

$$T_R(\lambda) \otimes_R \mathbb{C}_a \cong \mathrm{KZ}_{a(\mathbf{z})}(\Delta_{a(\mathbf{z})}(\lambda)).$$

Let K be the fraction field of R. By [GGOR, Theorem 2.19] the category $\mathcal{O}_{K,\mathbf{z},n}$ is split semisimple. In particular, the standard modules are simple. We have

$$\{T_K(\lambda), \lambda \in \mathcal{P}_{n,l}\} = \mathrm{Irr}(\mathscr{H}_{K,\mathbf{u},n}).$$

The Hecke algebra $\mathscr{H}_{K,\mathbf{u},n}$ is also split semisimple and we have

$$\{S_{K,\lambda}, \lambda \in \mathcal{P}_{n,l}\} = \mathrm{Irr}(\mathscr{H}_{K,\mathbf{u},n}),$$

see for example [A2, Corollary 13.9]. Thus there is a bijection $\varphi : \mathcal{P}_{n,l} \to \mathcal{P}_{n,l}$ such that $T_K(\lambda)$ is isomorphic to $S_{K,\varphi(\lambda)}$ for all λ. We claim that φ is identity. To see this, consider the algebra homomorphism $a_0 : R \to \mathbb{C}$ given by $\varpi \mapsto 0$. Then $\mathscr{H}_{a_0(\mathbf{u}),n}$ is canonically isomorphic to the group algebra $\mathbb{C}B_n(l)$, thus it is semi-simple. Let $\overline{K}$ be the algebraic closure of K. Let $\overline{R}$ be the integral closure of R in $\overline{K}$ and fix an extension $\overline{a}_0$ of a_0 to $\overline{R}$. By Tit's deformation theorem (see for example [CuR, Section 68A]), there is a bijection

$$\psi : \mathrm{Irr}(\mathscr{H}_{\overline{K},\mathbf{u},n}) \xrightarrow{\sim} \mathrm{Irr}(\mathscr{H}_{a_0(\mathbf{u}),n})$$

such that

$$\psi(T_{\overline{K}}(\lambda)) = T_{\overline{R}}(\lambda) \otimes_{\overline{R}} \mathbb{C}_{\overline{a}_0}, \quad \psi(S_{\overline{K},\lambda}) = S_{\overline{R},\lambda} \otimes_{\overline{R}} \mathbb{C}_{\overline{a}_0}.$$

By the definition of Specht modules we have $S_{\overline{R},\lambda}\otimes_{\overline{R}}\mathbb{C}_{\overline{a}_0}\cong\lambda$ as $\mathbb{C}B_n(l)$-modules. On the other hand, since $a_0(\mathbf{z})=0$, by (3.6) we have the following isomorphisms

$$\begin{aligned}T_{\overline{R}}(\lambda)\otimes_{\overline{R}}\mathbb{C}_{\overline{a}_0} &\cong T_R(\lambda)\otimes_R\mathbb{C}_{a_0}\\ &\cong \mathrm{KZ}_0(\Delta_0(\lambda))\\ &= \lambda.\end{aligned}$$

So $\psi(T_{\overline{K}}(\lambda))=\psi(S_{\overline{K},\lambda})$. Hence we have $T_{\overline{K}}(\lambda)\cong S_{\overline{K},\lambda}$. Since $T_{\overline{K}}(\lambda)=T_K(\lambda)\otimes_K\overline{K}$ is isomorphic to $S_{\overline{K},\varphi(\lambda)}=S_{K,\varphi(\lambda)}\otimes_K\overline{K}$, we deduce that $\varphi(\lambda)=\lambda$. The claim is proved.

Finally, let $\mathfrak{m}$ be the maximal ideal of R consisting of the functions vanishing at $\varpi=-1/el$. Let $\widehat{R}$ be the completion of R at $\mathfrak{m}$. It is a discrete valuation ring with residue field $\mathbb{C}$. Let $a_1:\widehat{R}\to\widehat{R}/\mathfrak{m}\widehat{R}=\mathbb{C}$ be the quotient map. We have $a_1(\mathbf{z})=\mathbf{h}$ and $a_1(\mathbf{u})=\mathbf{q}$. Let $\widehat{K}$ be the fraction field of $\widehat{R}$. Recall that the decomposition map is given by

$$d:K(\mathscr{H}_{\widehat{K},\mathbf{u},n})\to K(\mathscr{H}_{\mathbf{q},n}),\quad [M]\mapsto[L\otimes_{\widehat{R}}\mathbb{C}_{a_1}].$$

Here L is any free $\widehat{R}$-submodule of M such that $L\otimes_{\widehat{R}}\widehat{K}=M$. The choice of L does not affect the class $[L\otimes_{\widehat{R}}\mathbb{C}_{a_1}]$ in $K(\mathscr{H}_{\mathbf{q},n})$. See [A2, Section 13.3] for details on this map. Now, observe that we have

$$\begin{gathered}d([S_{\widehat{K},\lambda}])=[S_{\widehat{R},\lambda}\otimes_{\widehat{R}}\mathbb{C}_{a_1}]=[S_\lambda],\\ d([T_{\widehat{K}}(\lambda)])=[T_{\widehat{R}}(\lambda)\otimes_{\widehat{R}}\mathbb{C}_{a_1}]=[\mathrm{KZ}(\Delta(\lambda))].\end{gathered}$$

Since $\widehat{K}$ is an extension of K, by the last paragraph we have $[S_{\widehat{K},\lambda}]=[T_{\widehat{K}}(\lambda)]$. We deduce that $[\mathrm{KZ}(\Delta(\lambda))]=[S_\lambda]$. □

4. i-Restriction and i-Induction

We define in this section the i-restriction and i-induction functors for the cyclotomic rational DAHA's. This is done in parallel with the Hecke algebra case.

4.1. Let us recall the definition of the i-restriction and i-induction functors for $\mathscr{H}_{\mathbf{q},n}$. First define the Jucy-Murphy elements $J_0,\ldots,J_{n-1}$ in $\mathscr{H}_{\mathbf{q},n}$ by

$$J_0=T_0,\quad J_i=q^{-1}T_iJ_{i-1}T_i\quad\text{ for }1\leqslant i\leqslant n-1.$$

Write $Z(\mathscr{H}_{\mathbf{q},n})$ for the center of $\mathscr{H}_{\mathbf{q},n}$. For any symmetric polynomial σ of n variables the element $\sigma(J_0,\ldots,J_{n-1})$ belongs to $Z(\mathscr{H}_{\mathbf{q},n})$ (cf. [A2, Section 13.1]). In particular, if z is a formal variable the polynomial $C_n(z)=\prod_{i=0}^{n-1}(z-J_i)$ in $\mathscr{H}_{\mathbf{q},n}[z]$ has coefficients in $Z(\mathscr{H}_{\mathbf{q},n})$.

Now, for any $a(z)\in\mathbb{C}(z)$ let $P_{n,a(z)}$ be the exact endo-functor of the category $\mathscr{H}_{\mathbf{q},n}$-mod that maps an object M to the generalized eigenspace of $C_n(z)$ in M with the eigenvalue $a(z)$.

For any $i\in\mathbb{Z}/e\mathbb{Z}$ the i-restriction functor and i-induction functor

$$E_i(n)^{\mathscr{H}}:\mathscr{H}_{\mathbf{q},n}\text{-mod}\to\mathscr{H}_{\mathbf{q},n-1}\text{-mod},\quad F_i(n)^{\mathscr{H}}:\mathscr{H}_{\mathbf{q},n-1}\text{-mod}\to\mathscr{H}_{\mathbf{q},n}\text{-mod}$$

are defined as follows (cf. [A2, Definition 13.33]):

$$E_i(n)^{\mathscr{H}}=\bigoplus_{a(z)\in\mathbb{C}(z)}P_{n-1,a(z)/(z-q^i)}\circ E(n)^{\mathscr{H}}\circ P_{n,a(z)},$$

$$F_i(n)^{\mathscr{H}}=\bigoplus_{a(z)\in\mathbb{C}(z)}P_{n,a(z)(z-q^i)}\circ F(n)^{\mathscr{H}}\circ P_{n-1,a(z)}.$$

We will write

$$E_i^{\mathscr{H}}=\bigoplus_{n\geqslant1}E_i(n)^{\mathscr{H}},\quad F_i^{\mathscr{H}}=\bigoplus_{n\geqslant1}F_i(n)^{\mathscr{H}}.$$

They are endo-functors of $\mathscr{H}_{\mathbf{q},\mathbb{N}}$. For each $\lambda \in \mathcal{P}_{n,l}$ set

$$a_\lambda(z) = \prod_{v \in \Upsilon_\lambda} (z - q^{\mathrm{res}(v)}).$$

We recall some properties of these functors in the following proposition.

Proposition 4.1. *(1) The functors $E_i(n)^{\mathscr{H}}$, $F_i(n)^{\mathscr{H}}$ are exact. The functor $E_i(n)^{\mathscr{H}}$ is biadjoint to $F_i(n)^{\mathscr{H}}$.*

(2) For any $\lambda \in \mathcal{P}_{n,l}$ the element $C_n(z)$ has a unique eigenvalue on the Specht module S_λ. It is equal to $a_\lambda(z)$.

(3) We have

$$E_i(n)^{\mathscr{H}}([S_\lambda]) = \sum_{\mathrm{res}(\lambda/\mu)=i} [S_\mu], \qquad F_i(n)^{\mathscr{H}}([S_\lambda]) = \sum_{\mathrm{res}(\mu/\lambda)=i} [S_\mu].$$

(4) We have

$$E(n)^{\mathscr{H}} = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} E_i(n)^{\mathscr{H}}, \quad F(n)^{\mathscr{H}} = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} F_i(n)^{\mathscr{H}}.$$

Proof. Part (1) is obvious. See [A2, Theorem 13.21(2)] for (2) and [A2, Lemma 13.37] for (3). Part (4) follows from (3) and [A2, Lemma 13.32]. □

4.2. By Lemma 1.3(1) we have an algebra isomorphism

$$\gamma : Z(\mathcal{O}_{\mathbf{h},n}) \xrightarrow{\sim} Z(\mathscr{H}_{\mathbf{q},n}).$$

So there are unique elements $K_1, \ldots, K_n \in Z(\mathcal{O}_{\mathbf{h},n})$ such that the polynomial

$$D_n(z) = z^n + K_1 z^{n-1} + \cdots + K_n$$

maps to $C_n(z)$ by γ. Since the elements $K_1, \ldots, K_n$ act on simple modules by scalars and the category $\mathcal{O}_{\mathbf{h},n}$ is artinian, every module M in $\mathcal{O}_{\mathbf{h},n}$ is a direct sum of generalized eigenspaces of $D_n(z)$. For $a(z) \in \mathbb{C}(z)$ let $Q_{n,a(z)}$ be the exact endofunctor of $\mathcal{O}_{\mathbf{h},n}$ which maps an object M to the generalized eigenspace of $D_n(z)$ in M with the eigenvalue $a(z)$.

Definition 4.2. The *i-restriction* functor and the *i-induction* functor

$$E_i(n) : \mathcal{O}_{\mathbf{h},n} \to \mathcal{O}_{\mathbf{h},n-1}, \quad F_i(n) : \mathcal{O}_{\mathbf{h},n-1} \to \mathcal{O}_{\mathbf{h},n}$$

are given by

$$E_i(n) = \bigoplus_{a(z) \in \mathbb{C}(z)} Q_{n-1,a(z)/(z-q^i)} \circ E(n) \circ Q_{n,a(z)},$$

$$F_i(n) = \bigoplus_{a(z) \in \mathbb{C}(z)} Q_{n,a(z)(z-q^i)} \circ F(n) \circ Q_{n-1,a(z)}.$$

We will write

$$E_i = \bigoplus_{n \geqslant 1} E_i(n), \quad F_i = \bigoplus_{n \geqslant 1} F_i(n). \tag{4.1}$$

We have the following proposition.

Proposition 4.3. *For any $i \in \mathbb{Z}/e\mathbb{Z}$ there are isomorphisms of functors*

$$\mathrm{KZ} \circ E_i(n) \cong E_i^{\mathscr{H}}(n) \circ \mathrm{KZ}, \quad \mathrm{KZ} \circ F_i(n) \cong F_i^{\mathscr{H}}(n) \circ \mathrm{KZ}.$$

Proof. Since $\gamma(D_n(z)) = C_n(z)$, by Lemma 1.3(2) for any $a(z) \in \mathbb{C}(z)$ we have

$$\mathrm{KZ} \circ Q_{n,a(z)} \cong P_{n,a(z)} \circ \mathrm{KZ}.$$

So the proposition follows from Theorem 2.1 and Corollary 2.3. □

The next proposition is the DAHA version of Proposition 4.1.

Proposition 4.4. *(1) The functors $E_i(n)$, $F_i(n)$ are exact. The functor $E_i(n)$ is biadjoint to $F_i(n)$.*

(2) For any $\lambda \in \mathcal{P}_{n,l}$ the unique eigenvalue of $D_n(z)$ on the standard module $\Delta(\lambda)$ is $a_\lambda(z)$.

(3) We have the following equalities

$$E_i(n)([\Delta(\lambda)]) = \sum_{\mathrm{res}(\lambda/\mu)=i} [\Delta(\mu)], \qquad F_i(n)([\Delta(\lambda)]) = \sum_{\mathrm{res}(\mu/\lambda)=i} [\Delta(\mu)]. \tag{4.2}$$

(4) We have

$$E(n) = \bigoplus_{i\in\mathbb{Z}/e\mathbb{Z}} E_i(n), \quad F(n) = \bigoplus_{i\in\mathbb{Z}/e\mathbb{Z}} F_i(n).$$

Proof. (1) This is by construction and by Proposition 2.9.

(2) Since a standard module is indecomposable, the element $D_n(z)$ has a unique eigenvalue on $\Delta(\lambda)$. By Lemma 3.1 this eigenvalue is the same as the eigenvalue of $C_n(z)$ on S_λ.

(3) Let us prove the equality for $E_i(n)$. The Pieri rule for the group $B_n(l)$ together with Proposition 1.6(2) yields

$$E(n)([\Delta(\lambda)]) = \sum_{|\lambda/\mu|=1} [\Delta(\mu)], \quad F(n)([\Delta(\lambda)]) = \sum_{|\mu/\lambda|=1} [\Delta(\mu)]. \tag{4.3}$$

So we have

$$\begin{aligned}
E_i(n)([\Delta(\lambda)]) &= \bigoplus_{a(z)\in\mathbb{C}[z]} Q_{n-1,a(z)/(z-q^i)}(E(n)(Q_{n,a(z)}([\Delta(\lambda)]))) \\
&= Q_{n-1,a_\lambda(z)/(z-q^i)}(E(n)(Q_{n,a_\lambda(z)}([\Delta(\lambda)]))) \\
&= Q_{n-1,a_\lambda(z)/(z-q^i)}(E(n)([\Delta(\lambda)])) \\
&= Q_{n-1,a_\lambda(z)/(z-q^i)}(\sum_{|\lambda/\mu|=1} [\Delta(\mu)]) \\
&= \sum_{\mathrm{res}(\lambda/\mu)=i} [\Delta(\mu)].
\end{aligned}$$

The last equality follows from the fact that for any l-partition μ such that $|\lambda/\mu| = 1$ we have $a_\lambda(z) = a_\mu(z)(z - q^{\mathrm{res}(\lambda/\mu)})$. The proof for $F_i(n)$ is similar.

(4) It follows from part (3) and (4.3). $\square$

Corollary 4.5. *Under the isomorphism θ in (3.5) the operators E_i and F_i on $K(\mathcal{O}_{\mathbf{h},\mathbb{N}})$ go respectively to the operators e_i and f_i on $\mathcal{F}_{\mathbf{s}}$. When i runs over $\mathbb{Z}/e\mathbb{Z}$ they yield an action of $\widehat{\mathfrak{sl}}_e$ on $K(\mathcal{O}_{\mathbf{h},\mathbb{N}})$ such that θ is an isomorphism of $\widehat{\mathfrak{sl}}_e$-modules.*

Proof. This is clear from Proposition 4.4(3) and from (3.3). $\square$

5. $\widehat{\mathfrak{sl}}_e$-CATEGORIFICATION

In this section, we construct an $\widehat{\mathfrak{sl}}_e$-categorification on the category $\mathcal{O}_{\mathbf{h},\mathbb{N}}$ under some mild assumption on the parameter $\mathbf{h}$ (Theorem 5.1).

5.1. Recall that we put $q = \exp(\frac{2\pi\sqrt{-1}}{e})$ and P denotes the weight lattice. Let $\mathcal{C}$ be a $\mathbb{C}$-linear artinian abelian category. For any functor $F : \mathcal{C} \to \mathcal{C}$ and any $X \in \mathrm{End}(F)$, the generalized eigenspace of X acting on F with eigenvalue $a \in \mathbb{C}$ will be called the a-eigenspace of X in F. By [R2, Definition 5.29] an $\widehat{\mathfrak{sl}}_e$-*categorification* on $\mathcal{C}$ is the data of

(a) an adjoint pair (U, V) of exact functors $\mathcal{C} \to \mathcal{C}$,
(b) $X \in \mathrm{End}(U)$ and $T \in \mathrm{End}(U^2)$,

(c) a decomposition $\mathcal{C} = \bigoplus_{\tau \in P} \mathcal{C}_\tau$.

such that, set U_i (resp. V_i) to be the q^i-eigenspace of X in U (resp. in V)[1] for $i \in \mathbb{Z}/e\mathbb{Z}$, we have

(1) $U = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} U_i$,

(2) the endomorphisms X and T satisfy

$$\begin{aligned}(1_U T) \circ (T 1_U) \circ (1_U T) = (T 1_U) \circ (1_U T) \circ (T 1_U),\\ (T + 1_{U^2}) \circ (T - q 1_{U^2}) = 0,\\ T \circ (1_U X) \circ T = q X 1_U,\end{aligned} \tag{5.1}$$

(3) the action of $e_i = U_i$, $f_i = V_i$ on $K(\mathcal{C})$ with i running over $\mathbb{Z}/e\mathbb{Z}$ gives an integrable representation of $\widehat{\mathfrak{sl}}_e$.

(4) $U_i(\mathcal{C}_\tau) \subset \mathcal{C}_{\tau+\alpha_i}$ and $V_i(\mathcal{C}_\tau) \subset \mathcal{C}_{\tau-\alpha_i}$,

(5) V is isomorphic to a left adjoint of U.

5.2. We construct a $\widehat{\mathfrak{sl}}_e$-categorification on $\mathcal{O}_{\mathbf{h},\mathbb{N}}$ in the following way. The adjoint pair will be given by (E, F). To construct the part (b) of the data we need to go back to Hecke algebras. Following [CR, Section 7.2.2] let $X^{\mathscr{H}}$ be the endomorphism of $E^{\mathscr{H}}$ given on $E^{\mathscr{H}}(n)$ as the multiplication by the Jucy-Murphy element J_{n-1}. Let $T^{\mathscr{H}}$ be the endomorphism of $(E^{\mathscr{H}})^2$ given on $E^{\mathscr{H}}(n) \circ E^{\mathscr{H}}(n-1)$ as the multiplication by the element T_{n-1} in $\mathscr{H}_{\mathbf{q},n}$. The endomorphisms $X^{\mathscr{H}}$ and $T^{\mathscr{H}}$ satisfy the relations (5.1). Moreover the q^i-eigenspace of $X^{\mathscr{H}}$ in $E^{\mathscr{H}}$ and $F^{\mathscr{H}}$ gives respectively the i-restriction functor $E_i^{\mathscr{H}}$ and the i-induction functor $F_i^{\mathscr{H}}$ for any $i \in \mathbb{Z}/e\mathbb{Z}$.

By Theorem 2.1 we have an isomorphism $\mathrm{KZ} \circ E \cong E^{\mathscr{H}} \circ \mathrm{KZ}$. This yields an isomorphism

$$\mathrm{End}(\mathrm{KZ} \circ E) \cong \mathrm{End}(E^{\mathscr{H}} \circ \mathrm{KZ}).$$

By Proposition 1.9(1) the functor E maps projective objects to projective ones, so Lemma 2.4 applied to $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}_{\mathbf{h},\mathbb{N}}$ and $K = L = E$ yields an isomorphism

$$\mathrm{End}(E) \cong \mathrm{End}(\mathrm{KZ} \circ E).$$

Composing it with the isomorphism above gives a ring isomorphism

$$\sigma_E : \mathrm{End}(E) \xrightarrow{\sim} \mathrm{End}(E^{\mathscr{H}} \circ \mathrm{KZ}). \tag{5.2}$$

Replacing E by E^2 we get another isomorphism

$$\sigma_{E^2} : \mathrm{End}(E^2) \xrightarrow{\sim} \mathrm{End}((E^{\mathscr{H}})^2 \circ \mathrm{KZ}).$$

The data of $X \in \mathrm{End}(E)$ and $T \in \mathrm{End}(E^2)$ in our $\widehat{\mathfrak{sl}}_e$-categorification on $\mathcal{O}_{\mathbf{h},\mathbb{N}}$ will be provided by

$$X = \sigma_E^{-1}(X^{\mathscr{H}} 1_{\mathrm{KZ}}), \quad T = \sigma_{E^2}^{-1}(T^{\mathscr{H}} 1_{\mathrm{KZ}}).$$

Finally, the part (c) of the data will be given by the block decomposition of the category $\mathcal{O}_{\mathbf{h},\mathbb{N}}$. Recall from [LM, Theorem 2.11] that the block decomposition of the category $\mathscr{H}_{\mathbf{q},\mathbb{N}}$-mod yields

$$\mathscr{H}_{\mathbf{q},\mathbb{N}}\text{-mod} = \bigoplus_{\tau \in P} (\mathscr{H}_{\mathbf{q},\mathbb{N}}\text{-mod})_\tau,$$

where $(\mathscr{H}_{\mathbf{q},\mathbb{N}}\text{-mod})_\tau$ is the subcategory generated by the composition factors of the Specht modules S_λ with λ running over l-partitions of weight τ. By convention $(\mathscr{H}_{\mathbf{q},\mathbb{N}}\text{-mod})_\tau$ is zero if such λ does not exist. By Lemma 1.3 the functor KZ induces

[1] Here X acts on V via the isomorphism $\mathrm{End}(U) \cong \mathrm{End}(V)^{op}$ given by adjunction, see [CR, Section 4.1.2] for the precise definition.

a bijection between the blocks of the category $\mathcal{O}_{\mathbf{h},\mathbb{N}}$ and the blocks of $\mathscr{H}_{\mathbf{q},\mathbb{N}}$-mod. So the block decomposition of $\mathcal{O}_{\mathbf{h},\mathbb{N}}$ is

$$\mathcal{O}_{\mathbf{h},\mathbb{N}} = \bigoplus_{\tau\in P} (\mathcal{O}_{\mathbf{h},\mathbb{N}})_\tau,$$

where $(\mathcal{O}_{\mathbf{h},\mathbb{N}})_\tau$ is the block corresponding to $(\mathscr{H}_{\mathbf{q},\mathbb{N}}\text{-mod})_\tau$ via KZ.

5.3. Now we prove the following theorem.

Theorem 5.1. *The data of*

(a) *the adjoint pair* (E, F)*,*
(b) *the endomorphisms* $X \in \mathrm{End}(E)$*,* $T \in \mathrm{End}(E^2)$*,*
(c) *the decomposition* $\mathcal{O}_{\mathbf{h},\mathbb{N}} = \bigoplus_{\tau\in P}(\mathcal{O}_{\mathbf{h},\mathbb{N}})_\tau$

is a $\widehat{\mathfrak{sl}}_e$*-categorification on* $\mathcal{O}_{\mathbf{h},\mathbb{N}}$*.*

Proof. First, we prove that for any $i \in \mathbb{Z}/e\mathbb{Z}$ the q^i-generalized eigenspaces of X in E and F are respectively the i-restriction functor E_i and the i-induction functor F_i as defined in (4.1).

Recall from Proposition 4.1(4) and Proposition 4.4(4) that we have

$$E = \bigoplus_{i\in\mathbb{Z}/e\mathbb{Z}} E_i \quad \text{and} \quad E^{\mathscr{H}} = \bigoplus_{i\in\mathbb{Z}/e\mathbb{Z}} E_i^{\mathscr{H}}.$$

By the proof of Proposition 4.3 we see that any isomorphism

$$\mathrm{KZ}\circ E \cong E^{\mathscr{H}} \circ \mathrm{KZ}$$

restricts to an isomorphism $\mathrm{KZ}\circ E_i \cong E_i^{\mathscr{H}} \circ \mathrm{KZ}$ for each $i \in \mathbb{Z}/e\mathbb{Z}$. So the isomorphism σ_E in (5.2) maps $\mathrm{Hom}(E_i, E_j)$ to $\mathrm{Hom}(E_i^{\mathscr{H}} \circ \mathrm{KZ}, E_j^{\mathscr{H}} \circ \mathrm{KZ})$. Write

$$X = \sum_{i,j\in\mathbb{Z}/e\mathbb{Z}} X_{ij}, \quad X^{\mathscr{H}} 1_{\mathrm{KZ}} = \sum_{i,j\in\mathbb{Z}/e\mathbb{Z}} (X^{\mathscr{H}} 1_{\mathrm{KZ}})_{ij}$$

with $X_{ij} \in \mathrm{Hom}(E_i, E_j)$ and $(X^{\mathscr{H}} 1_{\mathrm{KZ}})_{ij} \in \mathrm{Hom}(E_i^{\mathscr{H}} \circ \mathrm{KZ}, E_j^{\mathscr{H}} \circ \mathrm{KZ})$. We have

$$\sigma_E(X_{ij}) = (X^{\mathscr{H}} 1_{\mathrm{KZ}})_{ij}.$$

Since $E_i^{\mathscr{H}}$ is the q^i-eigenspace of $X^{\mathscr{H}}$ in $E^{\mathscr{H}}$, we have $(X^{\mathscr{H}} 1_{\mathrm{KZ}})_{ij} = 0$ for $i \neq j$ and $(X^{\mathscr{H}} 1_{\mathrm{KZ}})_{ii} - q^i$ is nilpotent for any $i \in \mathbb{Z}/e\mathbb{Z}$. Since σ_E is an isomorphism of rings, this implies that $X_{ij} = 0$ and $X_{ii} - q^i$ is nilpotent in $\mathrm{End}(E)$. So E_i is the q^i-eigenspace of X in E. The fact that F_i is the q^i-eigenspace of X in F follows from adjunction.

Now, let us check the conditions (1)–(5):

(1) It is given by Proposition 4.4(4).

(2) Since $X^{\mathscr{H}}$ and $T^{\mathscr{H}}$ satisfy relations in (5.1), the endomorphisms X and T also satisfy them. Because these relations are preserved by ring homomorphisms.

(3) It follows from Corollary 4.5.

(4) By the definition of $(\mathcal{O}_{\mathbf{h},\mathbb{N}})_\tau$ and Lemma 3.1, the standard modules in $(\mathcal{O}_{\mathbf{h},\mathbb{N}})_\tau$ are all the $\Delta(\lambda)$ such that $\mathrm{wt}(\lambda) = \tau$. By (3.4) if μ is an l-partition such that $\mathrm{res}(\lambda/\mu) = i$ then $\mathrm{wt}(\mu) = \mathrm{wt}(\lambda) + \alpha_i$. Now, the result follows from (4.2).

(5) This is Proposition 2.9. $\square$

6. Crystals

Using the $\widehat{\mathfrak{sl}}_e$-categorification in Theorem 5.1 we construct a crystal on $\mathcal{O}_{\mathbf{h},\mathbb{N}}$ and prove that it coincides with the crystal of the Fock space $\mathcal{F}_{\mathbf{s}}$ (Theorem 6.3).

6.1. A *crystal* (or more precisely, an $\widehat{\mathfrak{sl}}_e$-crystal) is a set B together with maps

$$\mathrm{wt}: B \to P, \quad \tilde{e}_i, \tilde{f}_i : B \to B \sqcup \{0\}, \quad \epsilon_i, \varphi_i : B \to \mathbb{Z} \sqcup \{-\infty\},$$

such that

- $\varphi_i(b) = \epsilon_i(b) + \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle$,
- if $\tilde{e}_i b \in B$, then $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$, $\epsilon_i(\tilde{e}_i b) = \epsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$,
- if $\tilde{f}_i b \in B$, then $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$, $\epsilon_i(\tilde{f}_i b) = \epsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$,
- let $b, b' \in B$, then $\tilde{f}_i b = b'$ if and only if $\tilde{e}_i b' = b$,
- if $\varphi_i(b) = -\infty$, then $\tilde{e}_i b = 0$ and $\tilde{f}_i b = 0$.

Let V be an integrable $\widehat{\mathfrak{sl}}_e$-module. For any nonzero $v \in V$ and any $i \in \mathbb{Z}/e\mathbb{Z}$ we set

$$l_i(v) = \max\{l \in \mathbb{N} : e_i^l v \neq 0\}.$$

Write $l_i(0) = -\infty$. For $l \geqslant 0$ let

$$V_i^{<l} = \{v \in V : l_i(v) < l\}.$$

A weight basis of V is a basis B of V such that each element of B is a weight vector. Following A. Berenstein and D. Kazhdan (cf. [BK, Definition 5.30]), a *perfect basis* of V is a weight basis B together with maps $\tilde{e}_i, \tilde{f}_i : B \to B \sqcup \{0\}$ for $i \in \mathbb{Z}/e\mathbb{Z}$ such that

- for $b, b' \in B$ we have $\tilde{f}_i b = b'$ if and only if $\tilde{e}_i b' = b$,
- we have $\tilde{e}_i(b) \neq 0$ if and only if $e_i(b) \neq 0$,
- if $e_i(b) \neq 0$ then we have

$$e_i(b) \in \mathbb{C}^* \tilde{e}_i(b) + V_i^{<l_i(b)-1}. \tag{6.1}$$

We denote it by $(B, \tilde{e}_i, \tilde{f}_i)$. For such a basis let $\mathrm{wt}(b)$ be the weight of b, let $\epsilon_i(b) = l_i(b)$ and let

$$\varphi_i(b) = \epsilon_i(b) + \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle$$

for all $b \in B$. The data

$$(B, \mathrm{wt}, \tilde{e}_i, \tilde{f}_i, \epsilon_i, \varphi_i) \tag{6.2}$$

is a crystal. We will always attach this crystal structure to $(B, \tilde{e}_i, \tilde{f}_i)$. We call $b \in B$ a primitive element if $e_i(b) = 0$ for all $i \in \mathbb{Z}/e\mathbb{Z}$. Let B^+ be the set of primitive elements in B. Let V^+ be the vector space spanned by all the primitive vectors in V. The following lemma is [BK, Claim 5.32].

Lemma 6.1. *For any perfect basis $(B, \tilde{e}_i, \tilde{f}_i)$ the set B^+ is a basis of V^+.*

Proof. By definition we have $B^+ \subset V^+$. Given a vector $v \in V^+$, there exist $\zeta_1, \ldots, \zeta_r \in \mathbb{C}^*$ and distinct elements $b_1, \ldots, b_r \in B$ such that $v = \sum_{j=1}^r \zeta_j b_j$. For any $i \in \mathbb{Z}/e\mathbb{Z}$ let $l_i = \max\{l_i(b_j) : 1 \leqslant j \leqslant r\}$ and $J = \{j : l_i(b_j) = l_i,\ 1 \leqslant j \leqslant r\}$. Then by the third property of perfect basis there exist $\eta_j \in \mathbb{C}^*$ for $j \in J$ and a vector $w \in V^{<l_i-1}$ such that $0 = e_i(v) = \sum_{j \in J} \zeta_j \eta_j \tilde{e}_i(b_j) + w$. For distinct $j, j' \in J$, we have $b_j \neq b_{j'}$, so $\tilde{e}_i(b_j)$ and $\tilde{e}_i(b_{j'})$ are different unless they are zero. Moreover, since $l_i(\tilde{e}_i(b_j)) = l_i - 1$, the equality yields that $\tilde{e}_i(b_j) = 0$ for all $j \in J$. So $l_i = 0$. Hence $b_j \in B^+$ for $j = 1, \ldots, r$. $\square$

6.2. Given an $\widehat{\mathfrak{sl}}_e$-categorification on a $\mathbb{C}$-linear artinian abelian category $\mathcal{C}$ with the adjoint pair of endo-functors (U, V), $X \in \mathrm{End}(U)$ and $T \in \mathrm{End}(U^2)$, one can construct a perfect basis of $K(\mathcal{C})$ as follows. For $i \in \mathbb{Z}/e\mathbb{Z}$ let U_i, V_i be the q^i-eigenspaces of X in U and V. By definition, the action of X restricts to each U_i. One can prove that T also restricts to endomorphism of $(U_i)^2$, see for example the beginning of Section 7 in [CR]. It follows that the data (U_i, V_i, X, T) gives an $\mathfrak{sl}_2$-categorification on $\mathcal{C}$ in the sense of [CR, Section 5.21]. By [CR, Proposition 5.20] this implies that for any simple object L in $\mathcal{C}$, the object $\mathrm{head}(U_i(L))$ (resp. $\mathrm{soc}(V_i L)$) is simple unless it is zero.

Let $B_{\mathcal{C}}$ be the set of isomorphism classes of simple objects in $\mathcal{C}$. As part of the data of the $\widehat{\mathfrak{sl}}_e$-categorification, we have a decomposition $\mathcal{C} = \oplus_{\tau \in P} \mathcal{C}_\tau$. For a simple module $L \in \mathcal{C}_\tau$, the weight of $[L]$ in $K(\mathcal{C})$ is τ. Hence $B_{\mathcal{C}}$ is a weight basis of $K(\mathcal{C})$. Now for $i \in \mathbb{Z}/e\mathbb{Z}$ define the maps

$$\begin{aligned} \tilde{e}_i : \quad & B_{\mathcal{C}} \to B_{\mathcal{C}} \sqcup \{0\}, \qquad [L] \mapsto [\mathrm{head}(U_i L)], \\ \tilde{f}_i : \quad & B_{\mathcal{C}} \to B_{\mathcal{C}} \sqcup \{0\}, \qquad [L] \mapsto [\mathrm{soc}(V_i L)]. \end{aligned}$$

Proposition 6.2. *The data $(B_{\mathcal{C}}, \tilde{e}_i, \tilde{f}_i)$ is a perfect basis of $K(\mathcal{C})$.*

Proof. Fix $i \in \mathbb{Z}/e\mathbb{Z}$. Let us check the conditions in the definition in order:

- for two simple modules L, $L' \in \mathcal{C}$, we have $\tilde{e}_i([L]) = [L']$ if and only if $0 \neq \mathrm{Hom}(U_i L, L') = \mathrm{Hom}(L, V_i L')$, if and only if $\tilde{f}_i([L']) = [L]$.
- it follows from the fact that any non trivial module has a non trivial head.
- this is [CR, Proposition 5.20(d)].

□

6.3. Let $B_{\mathcal{F}_{\mathbf{s}}}$ be the set of l-partitions. In [JMMO] this set is given a crystal structure. We will call it the crystal of the Fock space $\mathcal{F}_{\mathbf{s}}$.

Theorem 6.3. *(1) The set*

$$B_{\mathcal{O}_{\mathbf{h},\mathbb{N}}} = \{[L(\lambda)] \in K(\mathcal{O}_{\mathbf{h},\mathbb{N}}) : \lambda \in \mathcal{P}_{n,l}, n \in \mathbb{N}\}$$

and the maps

$$\begin{aligned} \tilde{e}_i : \quad & B_{\mathcal{O}_{\mathbf{h},\mathbb{N}}} \to B_{\mathcal{O}_{\mathbf{h},\mathbb{N}}} \sqcup \{0\}, \qquad [L] \mapsto [\mathrm{head}(E_i L)], \\ \tilde{f}_i : \quad & B_{\mathcal{O}_{\mathbf{h},\mathbb{N}}} \to B_{\mathcal{O}_{\mathbf{h},\mathbb{N}}} \sqcup \{0\}, \qquad [L] \mapsto [\mathrm{soc}(F_i L)]. \end{aligned}$$

define a crystal structure on $B_{\mathcal{O}_{\mathbf{h},\mathbb{N}}}$.

(2) The crystal $B_{\mathcal{O}_{\mathbf{h},\mathbb{N}}}$ given by (1) is isomorphic to the crystal $B_{\mathcal{F}_{\mathbf{s}}}$.

Proof. (1) Applying Proposition 6.2 to the $\widehat{\mathfrak{sl}}_e$-categorification in Theorem 5.1 yields that $(B_{\mathcal{O}_{\mathbf{h},\mathbb{N}}}, \tilde{e}_i, \tilde{f}_i)$ is a perfect basis. So it defines a crystal structure on $B_{\mathcal{O}_{\mathbf{h},\mathbb{N}}}$ by (6.2).

(2) It is known that $B_{\mathcal{F}_{\mathbf{s}}}$ is a perfect basis of $\mathcal{F}_{\mathbf{s}}$. Identify the $\widehat{\mathfrak{sl}}_e$-modules $\mathcal{F}_{\mathbf{s}}$ and $K(\mathcal{O}_{\mathbf{h},\mathbb{N}})$. By Lemma 6.1 the set $B^+_{\mathcal{F}_{\mathbf{s}}}$ and $B^+_{\mathcal{O}_{\mathbf{h},\mathbb{N}}}$ are two weight bases of $\mathcal{F}^+_{\mathbf{s}}$. So there is a bijection $\psi : B^+_{\mathcal{F}_{\mathbf{s}}} \to B^+_{\mathcal{O}_{\mathbf{h},\mathbb{N}}}$ such that $\mathrm{wt}(b) = \mathrm{wt}(\psi(b))$. Since $\mathcal{F}_{\mathbf{s}}$ is a direct sum of highest weight simple $\widehat{\mathfrak{sl}}_e$-modules, this bijection extends to an automorphism ψ of the $\widehat{\mathfrak{sl}}_e$-module $\mathcal{F}_{\mathbf{s}}$. By [BK, Main Theorem 5.37] any automorphism of $\mathcal{F}_{\mathbf{s}}$ which maps $B^+_{\mathcal{F}_{\mathbf{s}}}$ to $B^+_{\mathcal{O}_{\mathbf{h},\mathbb{N}}}$ induces an isomorphism of crystals $B_{\mathcal{F}_{\mathbf{s}}} \cong B_{\mathcal{O}_{\mathbf{h},\mathbb{N}}}$. □

Remark 6.4. One can prove that if $n < e$ then a simple module $L \in \mathcal{O}_{\mathbf{h},n}$ has finite dimension over $\mathbb{C}$ if and only if the class $[L]$ is a primitive element in $B_{\mathcal{O}_{\mathbf{h},\mathbb{N}}}$. In the case $n = 1$, we have $B_n(l) = \mu_l$, the cyclic group, and the primitive elements in the crystal $B_{\mathcal{F}_{\mathbf{s}}}$ have explicit combinatorial descriptions. This yields another proof

of the classification of finite dimensional simple modules of $H_{\mathbf{h}}(\mu_l)$, which was first given by W. Crawley-Boevey and M. P. Holland. See type A case of [CH, Theorem 7.4].

Acknowledgements

I would like to thank my advisor E. Vasserot for suggesting me this problem, for his guidance and his patience. I would also like to thank I. Gordon and M. Martino for enlightening discussions. Independently, they also obtained the result on crystals in their recent joint work. I am grateful to D. Yang and I. Losev for pointing out some errors in the preliminary version of this paper.

References

[A1] S. Ariki, *On the decomposition numbers of the Hecke algebra of $G(m,1,n)$*, J. Math. Kyoto Univ. **36** (1996), no. 4, 789-808.

[A2] S. Ariki, *Representations of quantum algebras and combinatorics of Young tableaux*, Univ. Lecture Ser., Vol. **26**, Amer. Math. Soc., 2002.

[BE] R. Bezrukavnikov, P. Etingof, *Parabolic induction and restriction functors for rational Cherednik algebras*, Selecta Math. (N.S.) **14** (2009), no. 3-4, 397-425.

[BK] A. Berenstein, D. Kazhdan, *Geometric and unipotent crystals II: From unipotent bicrystals to crystal bases*, in “Quantum groups”, 13-88, Contemp. Math. **433**, Amer. Math. Soc., 2007.

[BMM] M. Broué, G. Malle, J. Michel, *Towards Spetses I*, Trans. Groups **4**, No. 2-3 (1999), 157-218

[BMR] M. Broué, G. Malle, R. Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, J. Reine Angew. Math. **500** (1998), 127-190.

[CH] W. Crawley-Boevey, M. P. Holland, *Noncommutative deformations of Kleinian singularities*, Duke Math. J. **92** (1998), no.3, 605-635.

[CR] J. Chuang, R. Rouquier, *Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification*, Ann. of Math. (2)**167** (2008), no.1, 245-298.

[CuR] C.W. Curtis, I. Reiner, *Methods of representation theory -with application to finite groups and orders, volume II*, John Wiley & Sons, Inc., New York, 1981.

[EG] P. Etingof, V. Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphisms*, Invent. Math **147** (2002), 243-348.

[GGOR] V. Ginzburg, N. Guay, E. Opdam, R. Rouquier, *On the category $\mathcal{O}$ for rational Cherednik algebras*, Invent. Math. **154** (2003), no.3, 617-651.

[JMMO] M. Jimbo, K. Misra, T. Miwa, M. Okado, *Combinatorics of representations of $U_q(\widehat{\mathfrak{sl}}_n)$ at $q=0$*, Comm. Math. Phys. **136** (1991) 543-566.

[KK] M. Kashiwara, T. Kawai, *On holonomic systems of microdifferential equations III: Systems with regular singularities*, Publ. Res. Inst. Math. Sci. **17**-3(1981), 813-979.

[LM] S. Lyle, A. Mathas, *Blocks of cyclotomic Hecke algebras*, Adv. Math. **216**(2007), no.2, 854-878.

[R1] R. Rouquier, *q-Schur algebras and complex reflection groups*, Mosc. Math. J. **8** (2008), no. 1, 119-158.

[R2] R. Rouquier, *2-Kac-Moody algebras*, preprint, arXiv: 0812.5023

[U] D. Uglov *Canonical bases of higher-level q-deformed Fock spaces and Kazhdan-Lusztig polynomials*, in "Physical combinatorics (Kyoto, 1999)", 249-299, Birkhäuser, Boston, 2000.

[Y] X. Yvonne, *A conjecture for q-decomposition matrices of cyclotomic v-Schur algebras.* J. Algebra **304** (2006), no. 1, 419-456.

Université Paris 7, Institut de Mathématiques de Jussieu, Théorie des groupes et des représentations, Case 7012, 2 place Jussieu, 75251 Paris Cedex 05, France.

E-mail address: shan@math.jussieu.fr